Algebraic cryptanalysis and multivariate cryptography

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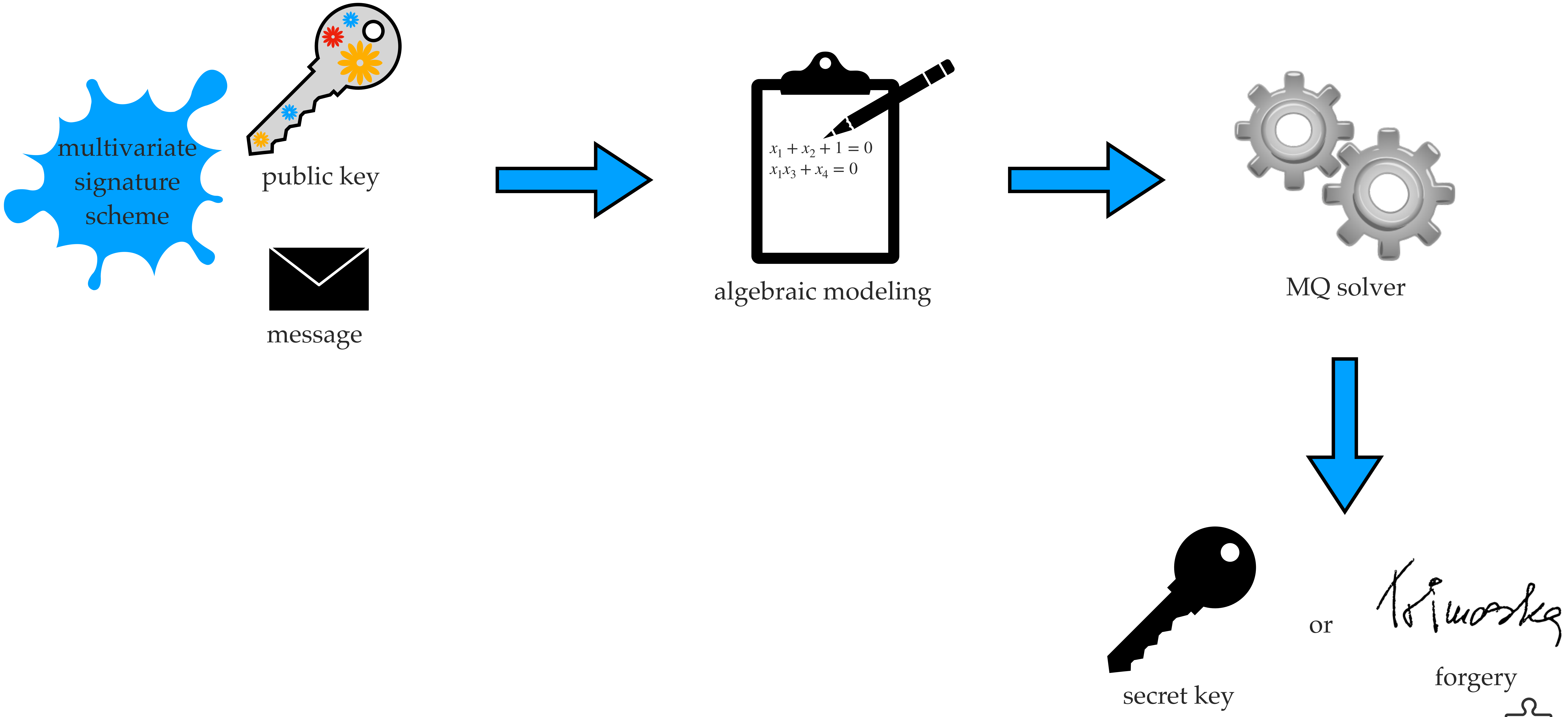
TU/e

*Animated version at: https://mtrimoska.com/slides/PQCSA_cryptanalysis/

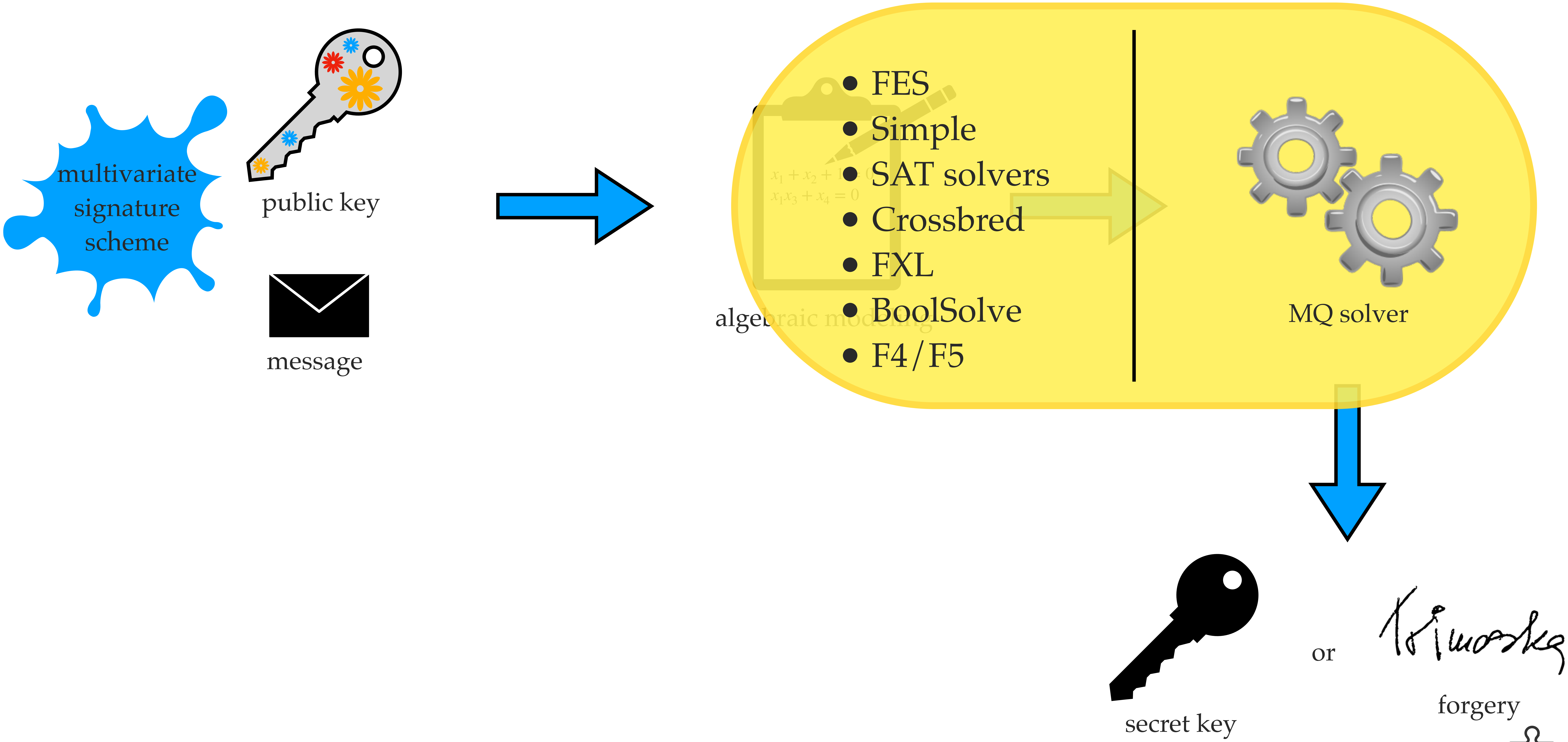
Algebraic cryptanalysis

A type of cryptanalytic methods where the problem of finding the secret key (or any attack goal) is reduced to the problem of finding a solution to a nonlinear multivariate polynomial system of equations.

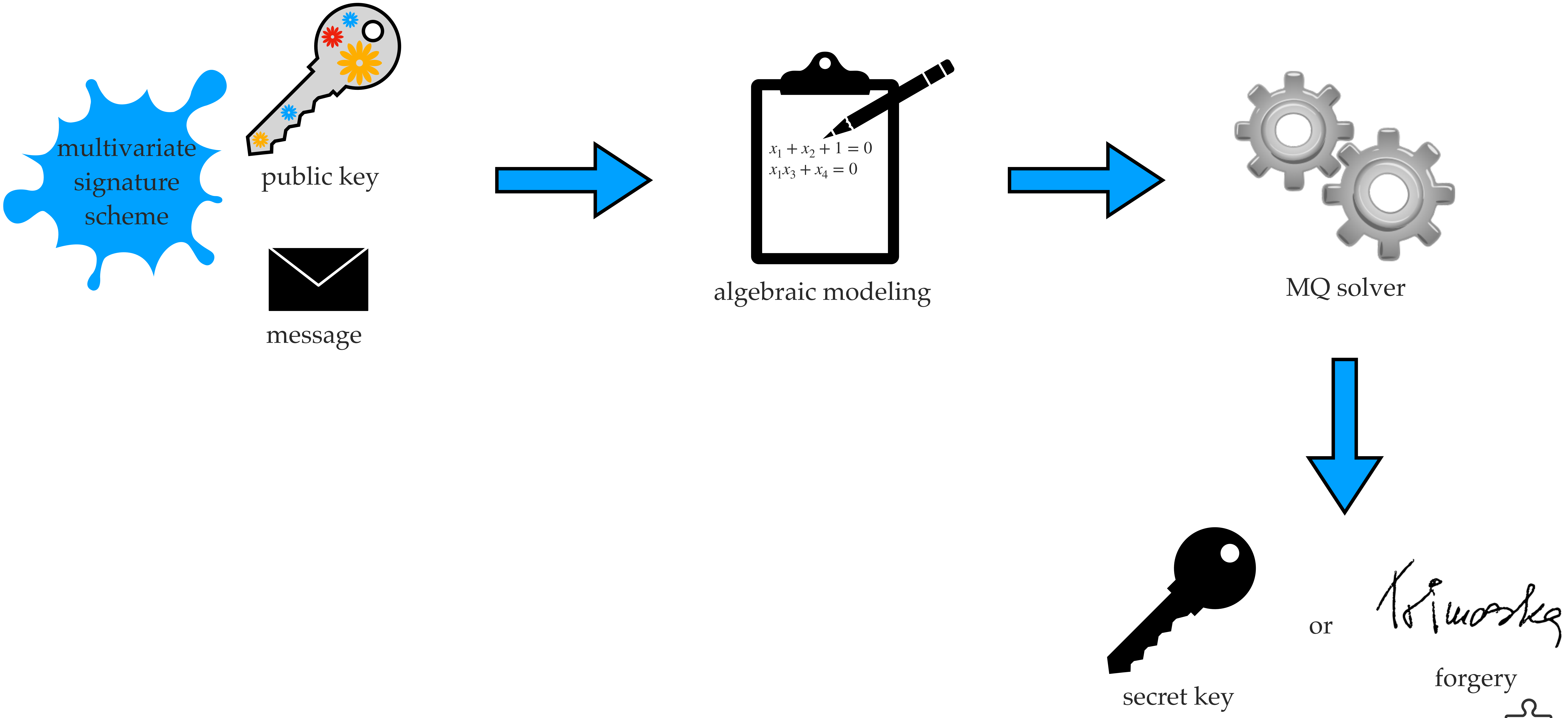
Algebraic cryptanalysis



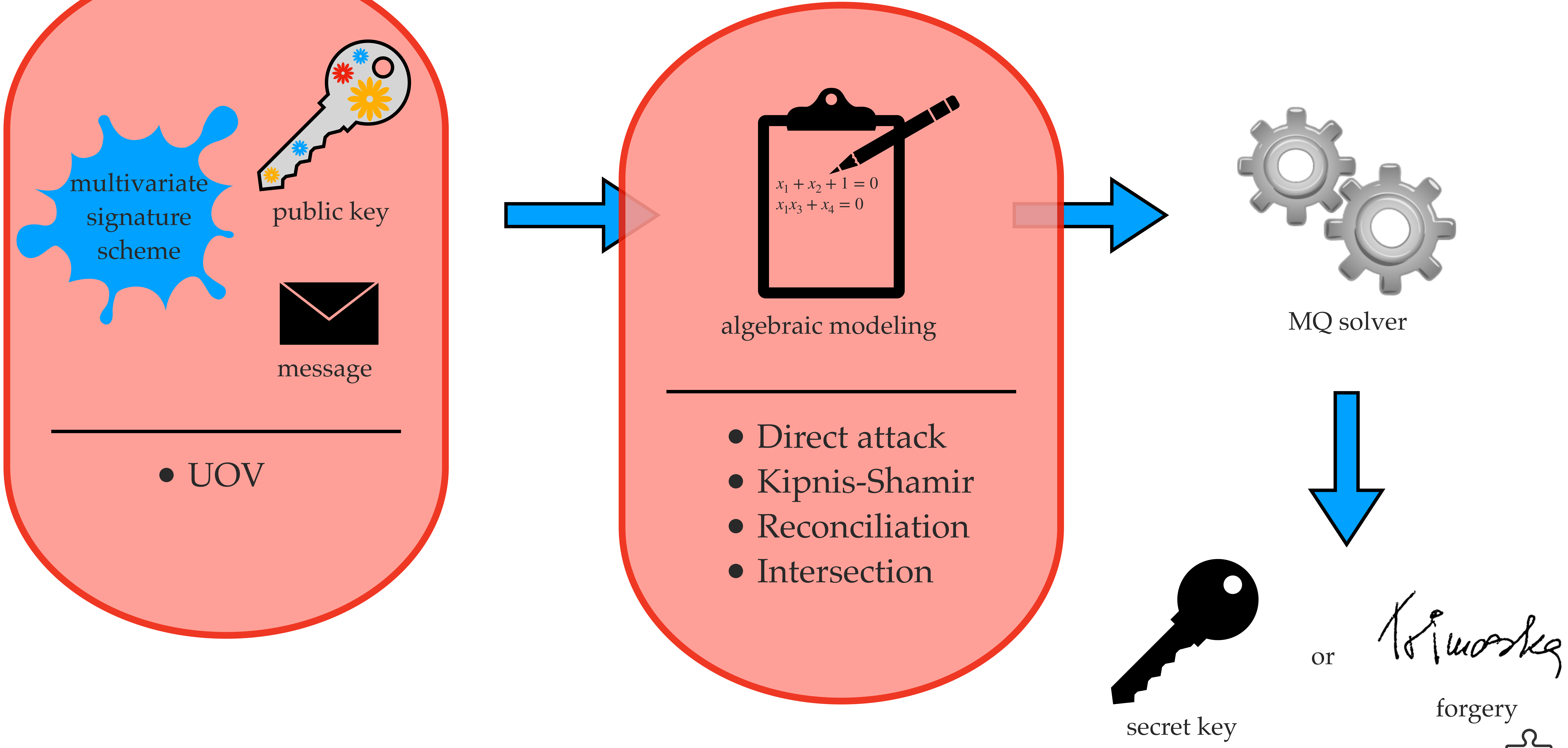
Algebraic cryptanalysis



Algebraic cryptanalysis



Algebraic cryptanalysis



The MQ problem (recall)

The MQ problem

Given m multivariate quadratic polynomials $f_1, \ldots, f_m$ of n variables over a finite field $\mathbb{F}_q$, find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f_1(\mathbf{x}) = \ldots = f_m(\mathbf{x}) = 0$.

Example.

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$

$$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$$

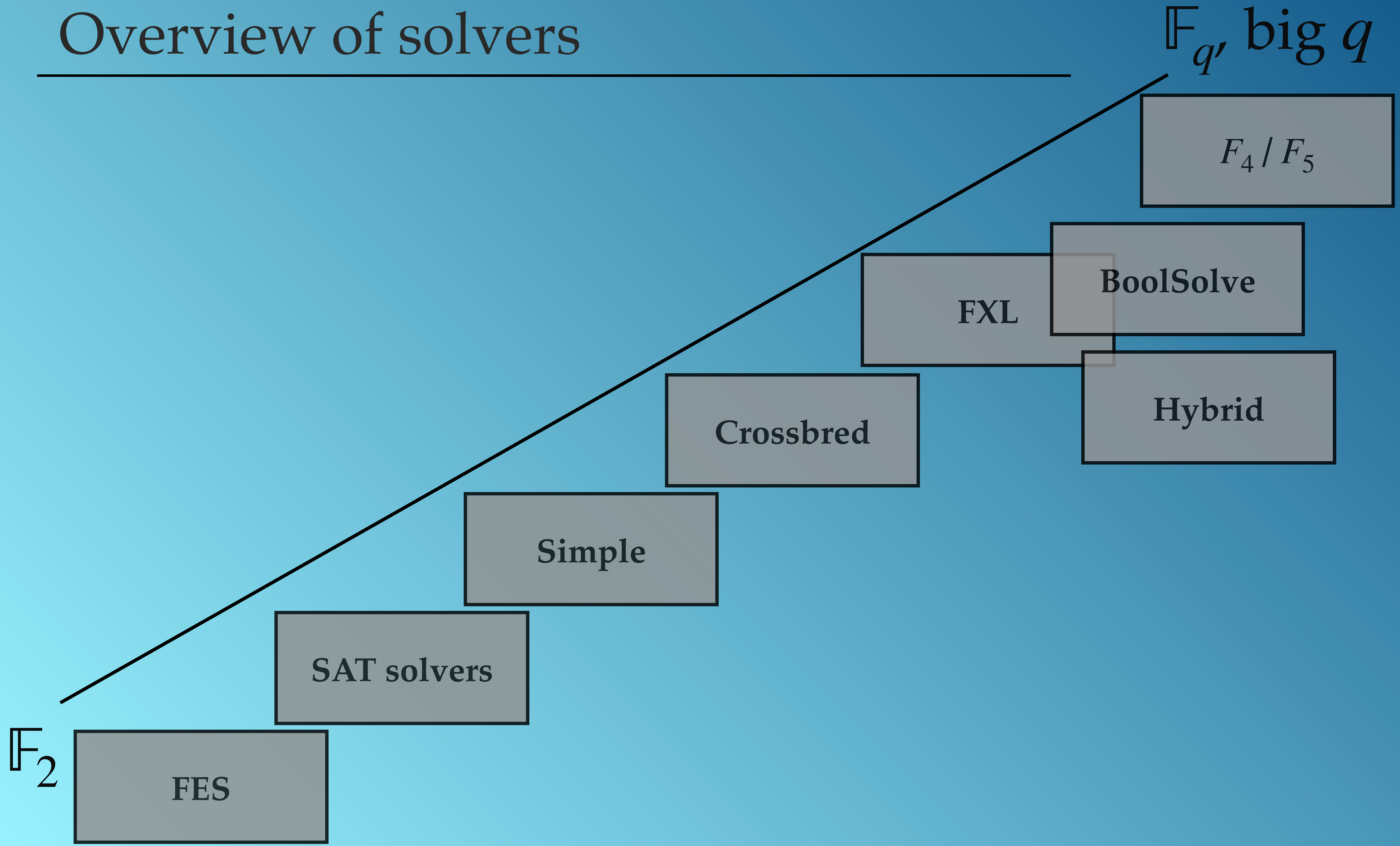
$$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$$

$$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$$

$$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$$

$$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$$

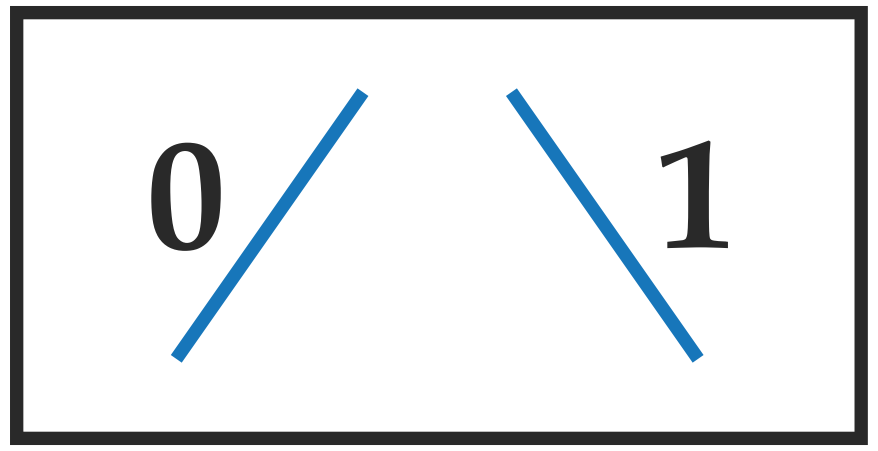
Overview of solvers

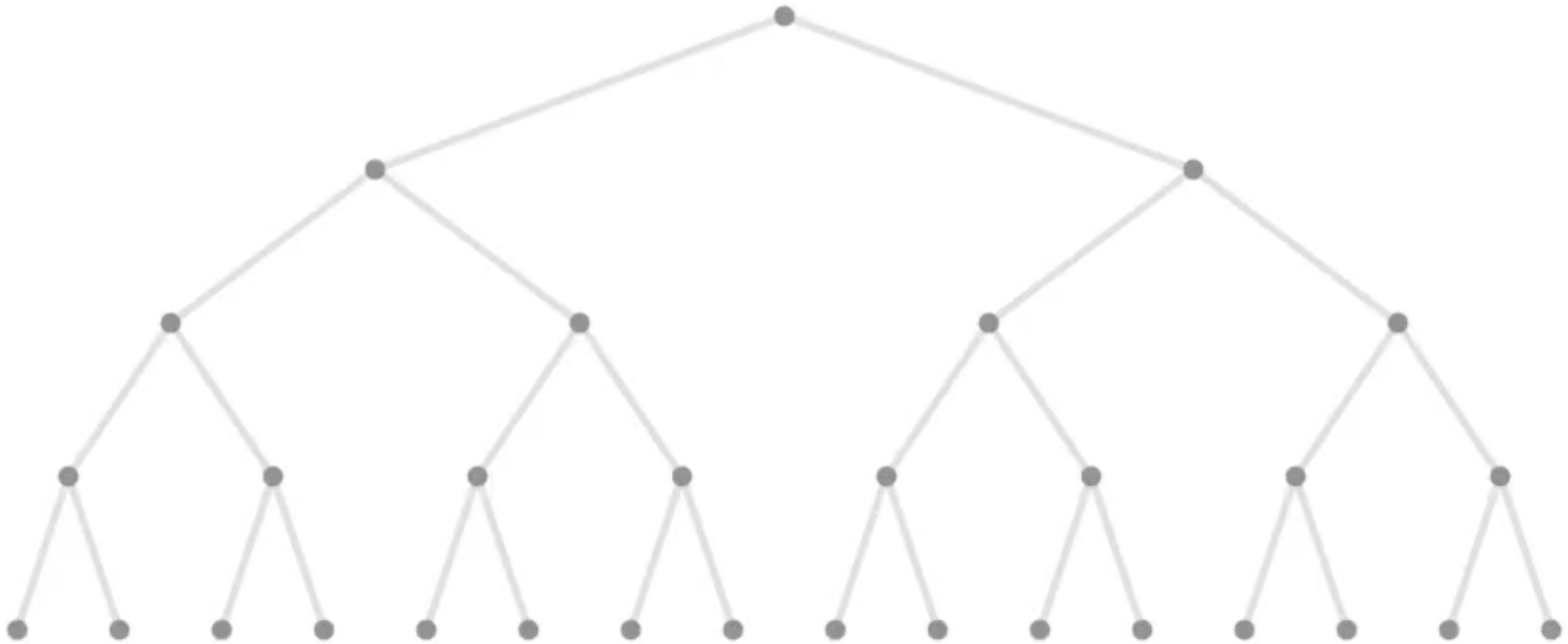


(Fast) Exhaustive Search

[Bouillaguet, Chen, Cheng, Chou, Niederhagen, Shamir, Yang, 2010]

Exhaustive Search





Binary search tree

$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$

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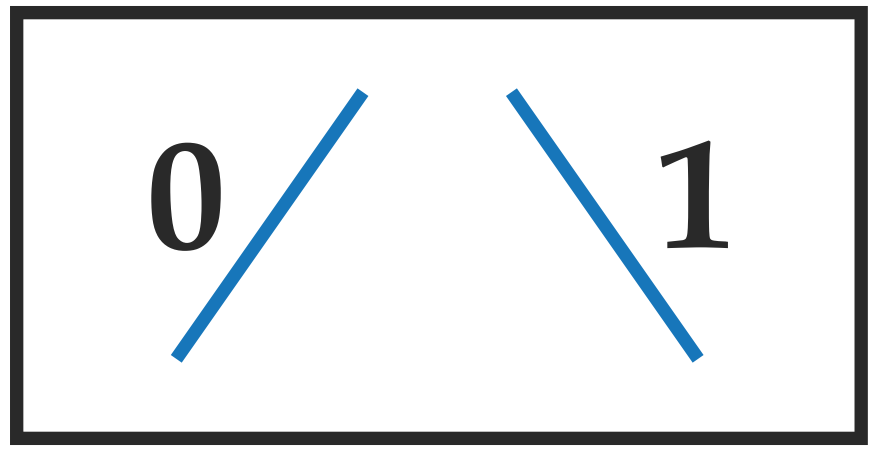

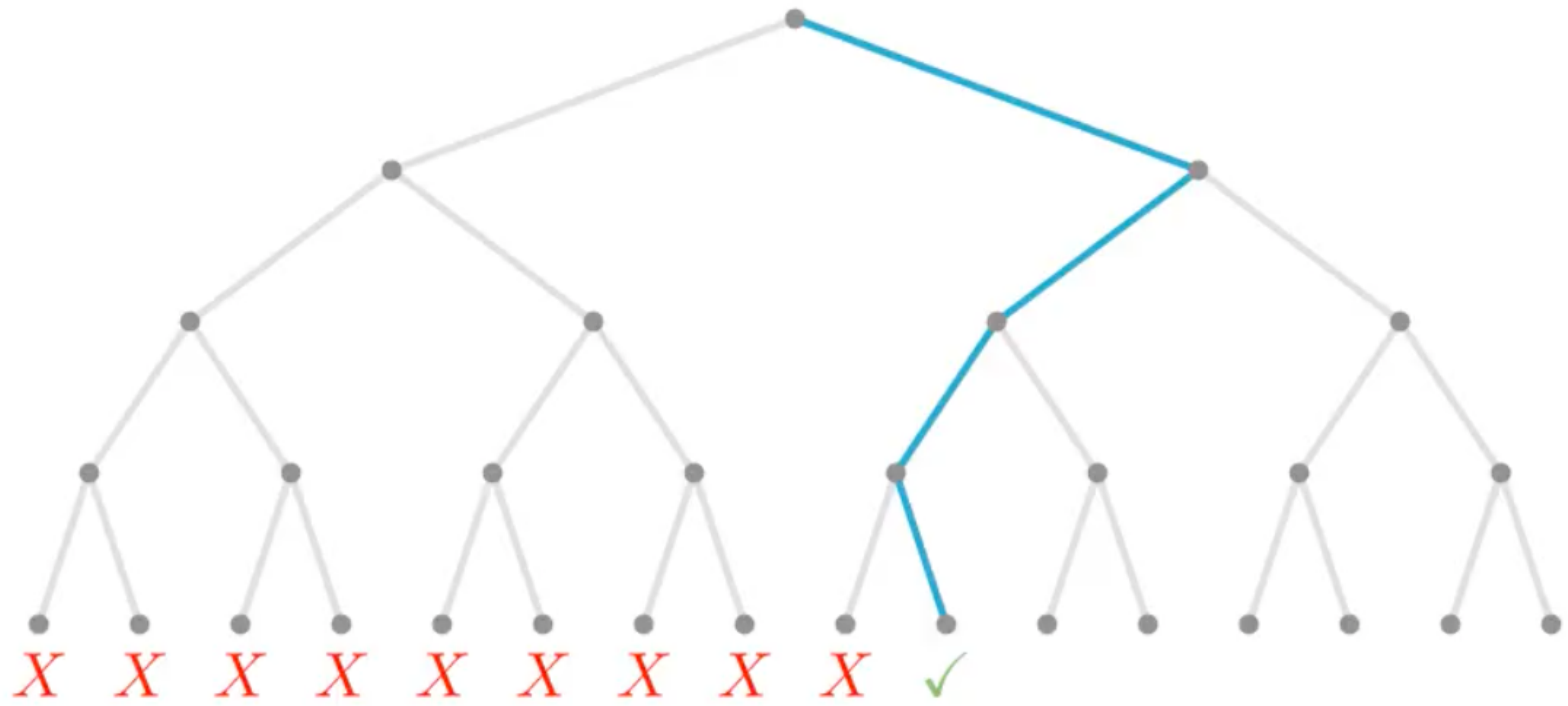

Binary search tree

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

$$0 \cdot 0 + 0 \cdot 1 + 1 + 0 + 1 = 0$$

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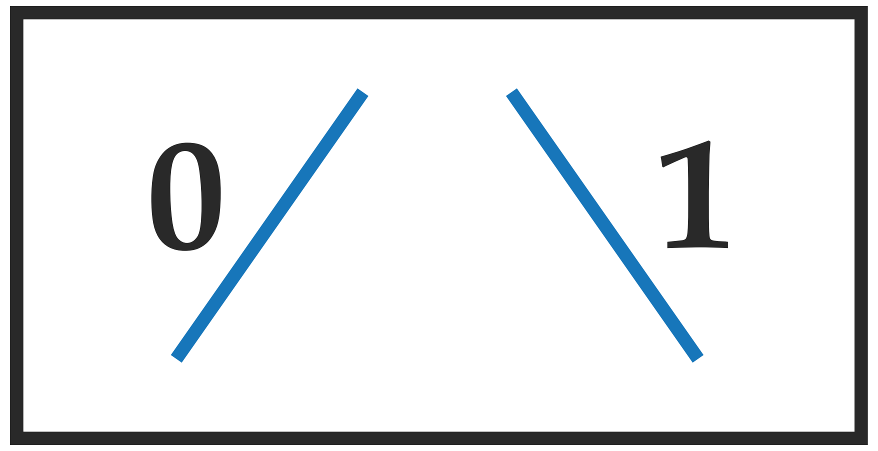
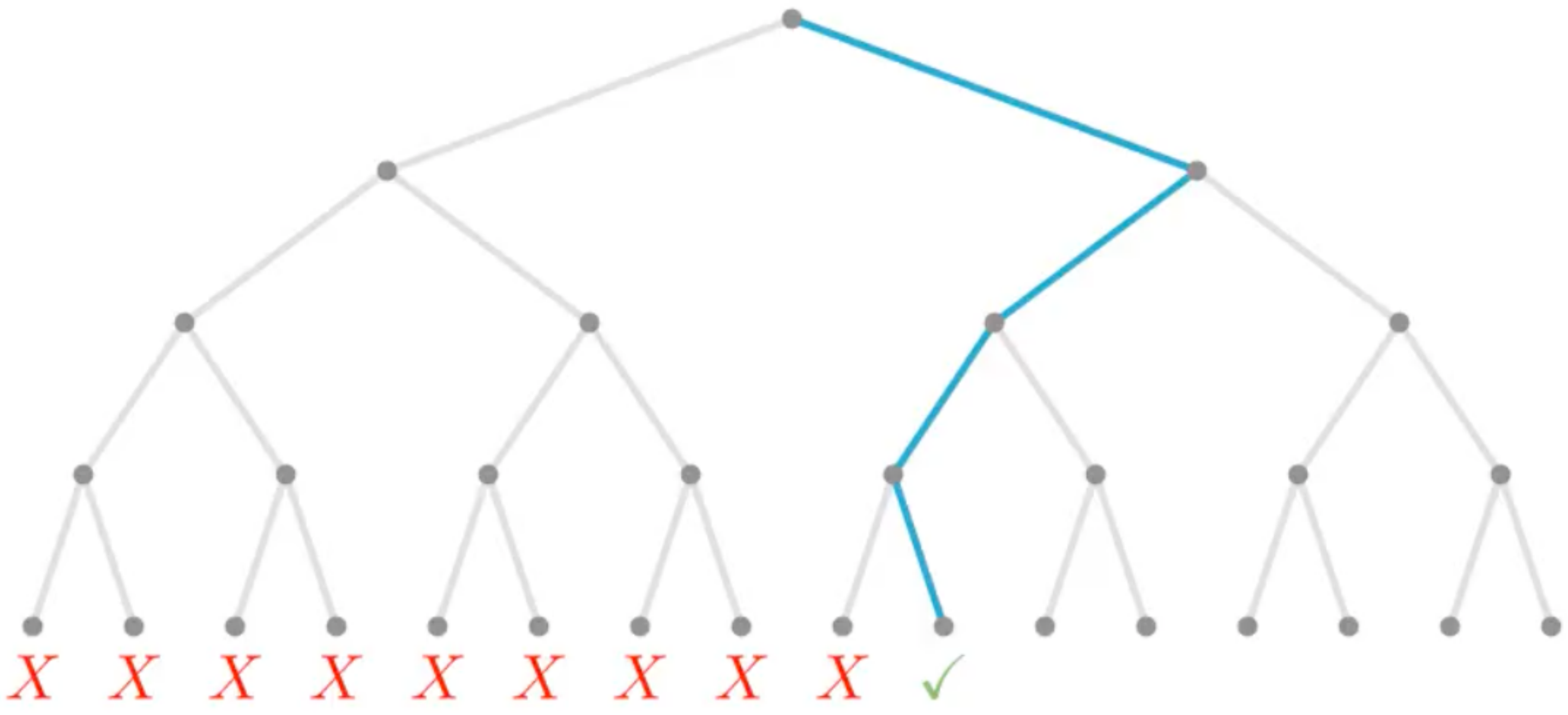


Worst-case complexity: $\mathcal{O}(2^n)$



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Fast Exhaustive Search

* The libFES solver

Gray code

- An ordering of the binary system where two successive values differ in only one bit.

Example. $n = 4$

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000

Fast Exhaustive Search

Gray code

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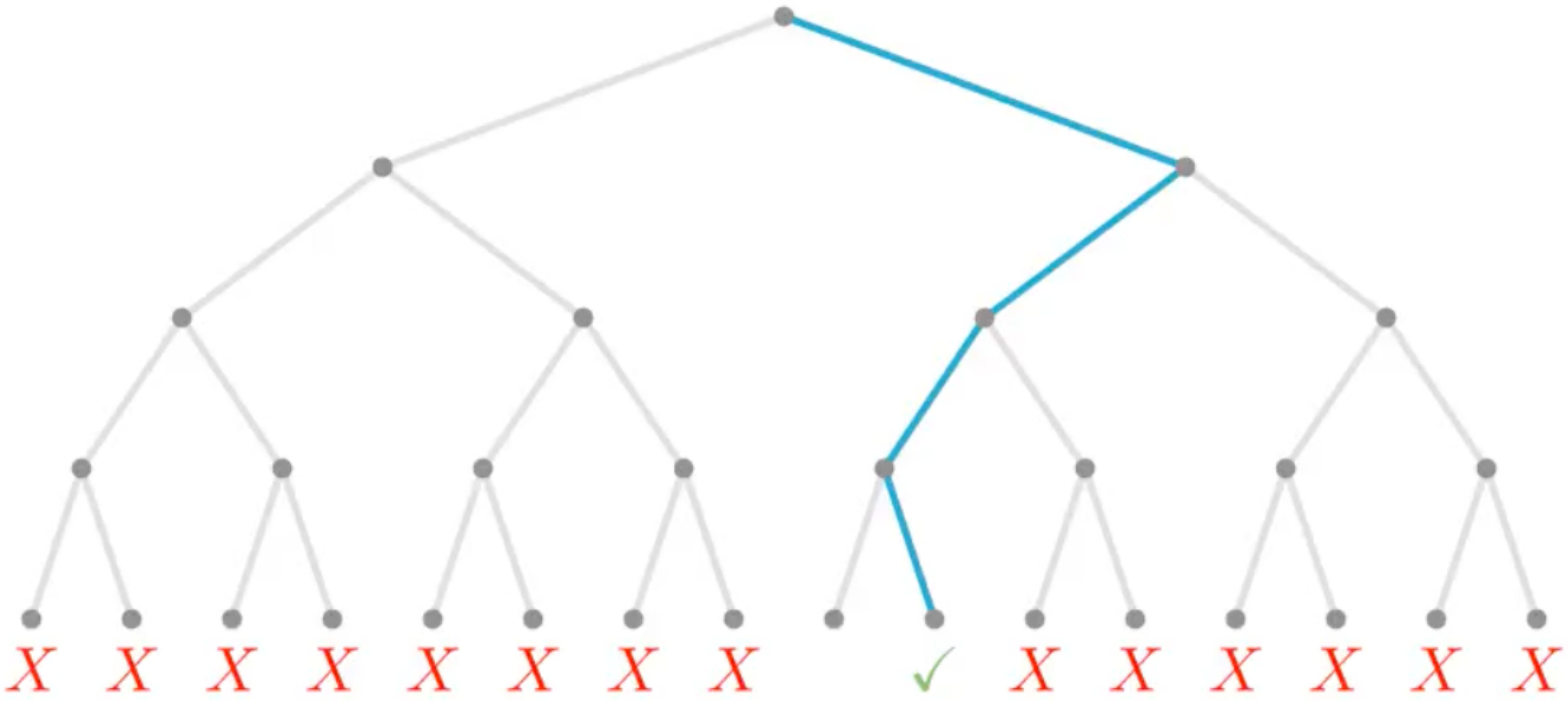

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

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Fast Exhaustive Search

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! But, it differs from the depth-first traversal in the polynomial factors

Gray code

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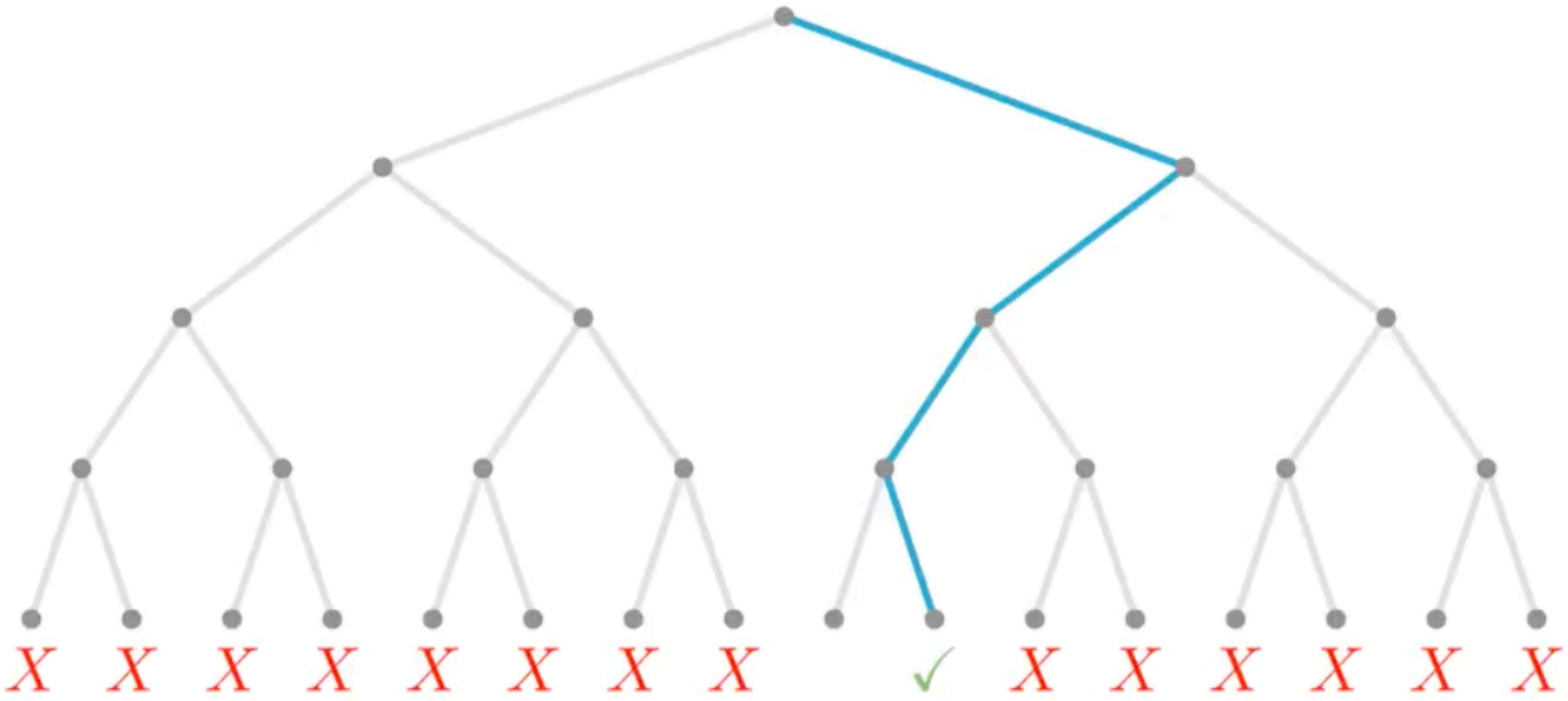

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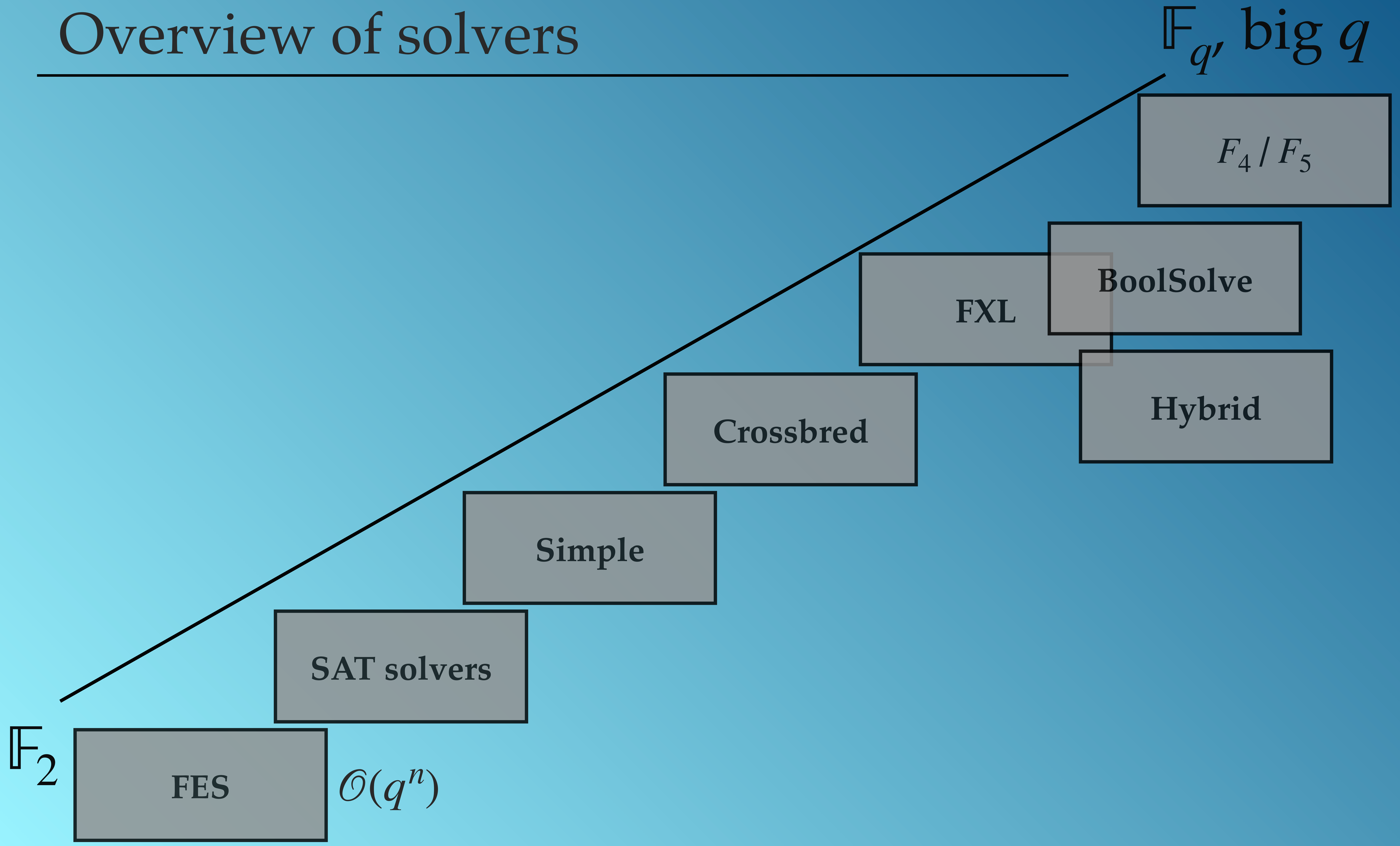
Overview of solvers



SAT solvers

CryptoMiniSat [Soos, Nohl, Castelluccia, 2009], WDSat [T., Dequen, Ionica, 2020]

Simple algorithm

[Bouillaguet, Delaplace, T., 2021]

(SAT solvers)

- Propositional formula in Conjunctive Normal Form (CNF): a conjunction of clauses where each clause is a disjunction of literals and where each literal is a variable or a negated variable.

Example.

$$(x_1 \lor \neg x_2) \land$$
$$(x_2 \lor x_3 \lor x_4) \land$$
$$(\neg x_1 \lor x_4)$$

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The SATisfiability problem

Given a propositional formula, determine whether there exists an interpretation (assignment of all variables) such that the formula is satisfied (evaluates to TRUE).

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Example.

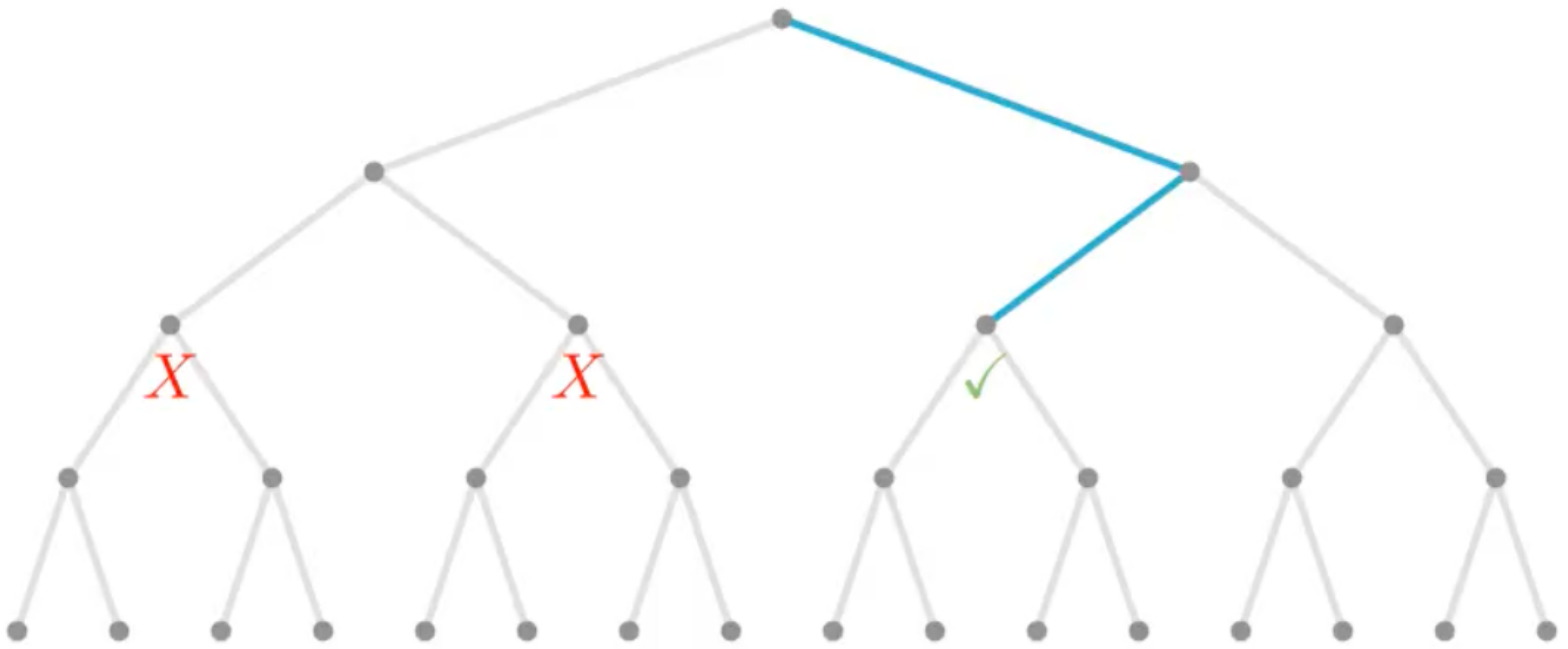
$$
\begin{aligned}
&(x_1 \vee \neg x_2) \wedge \\
&(x_2 \vee x_3 \vee x_4) \wedge \\
&(\neg x_1 \vee x_4)
\end{aligned}
$$

The SATisfiability problem

Given a propositional formula, determine whether there exists an interpretation (assignment of all variables) such that the formula is satisfied (evaluates to TRUE).

SAT solver: a tool for solving the SAT problem.

Partial assignment and conflicts



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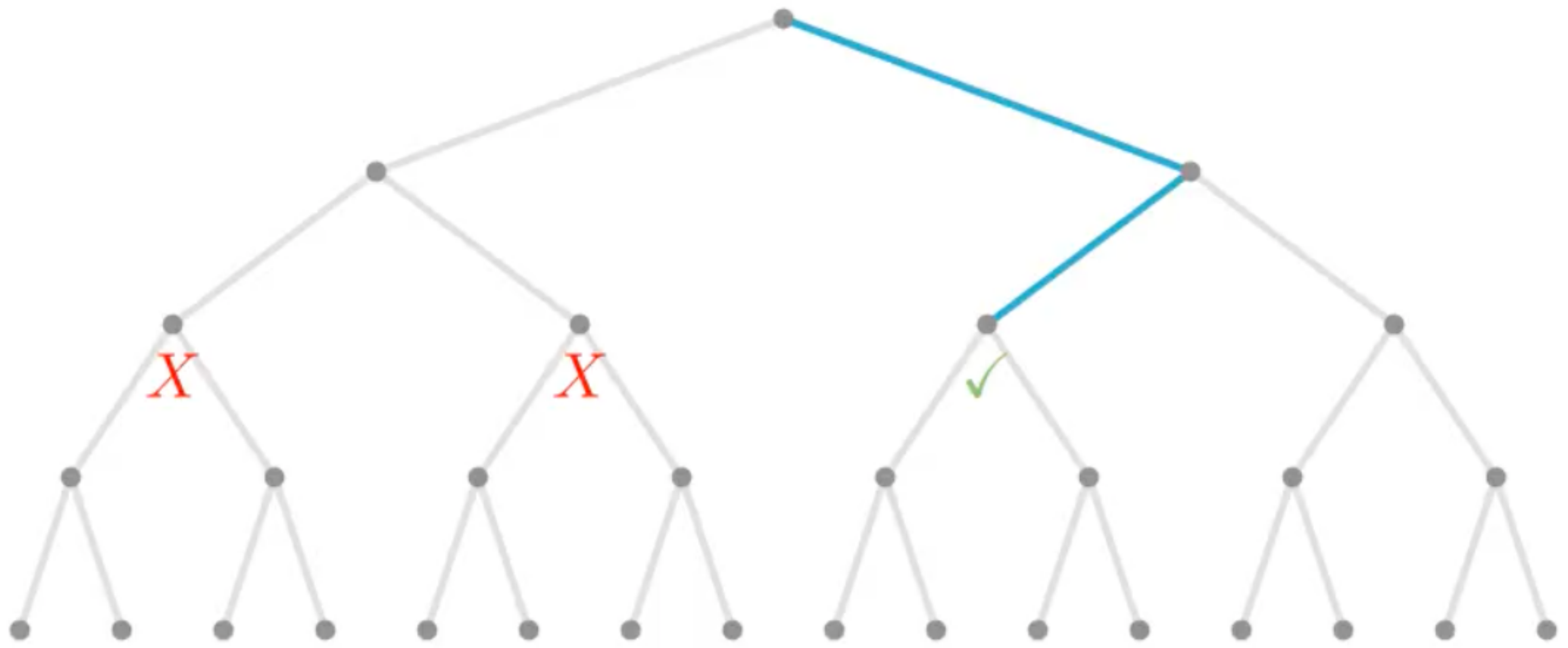
Partial assignment and conflicts

Which (portion of) branches are missing ??



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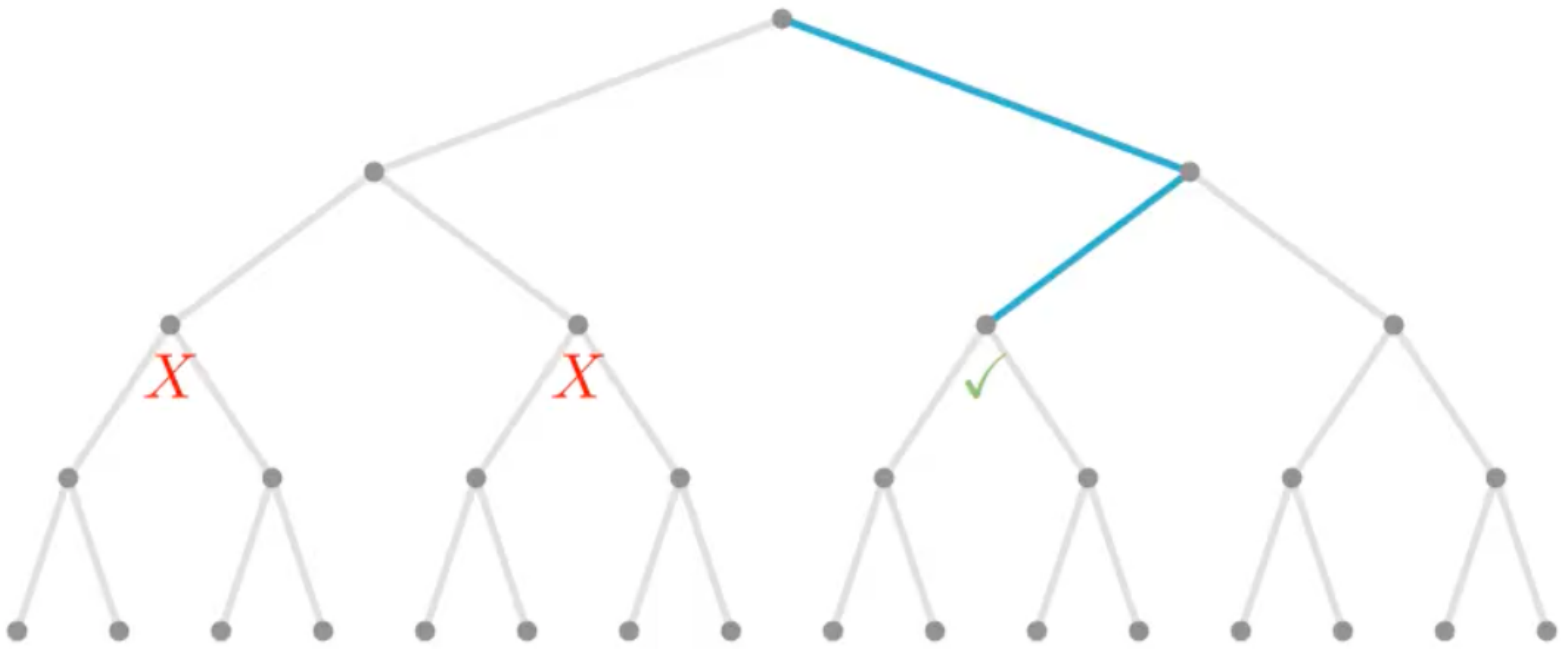

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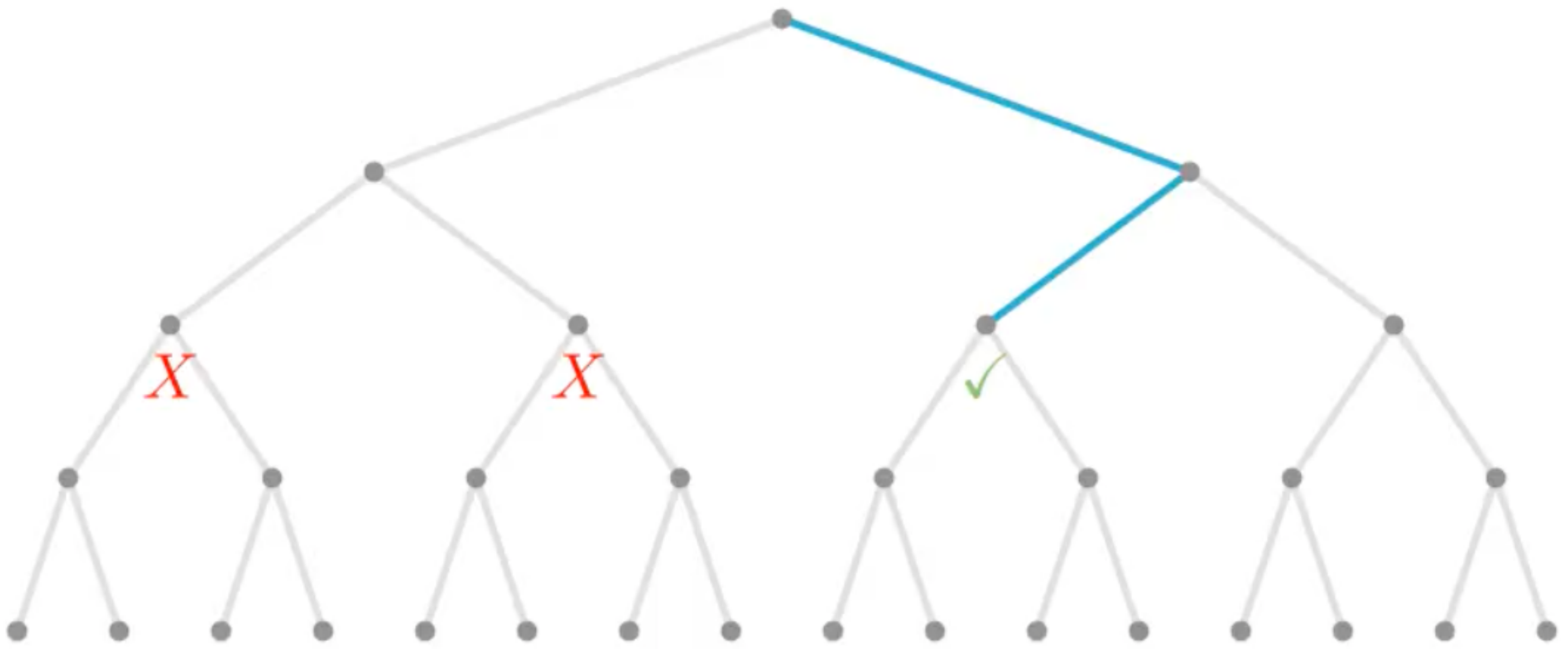

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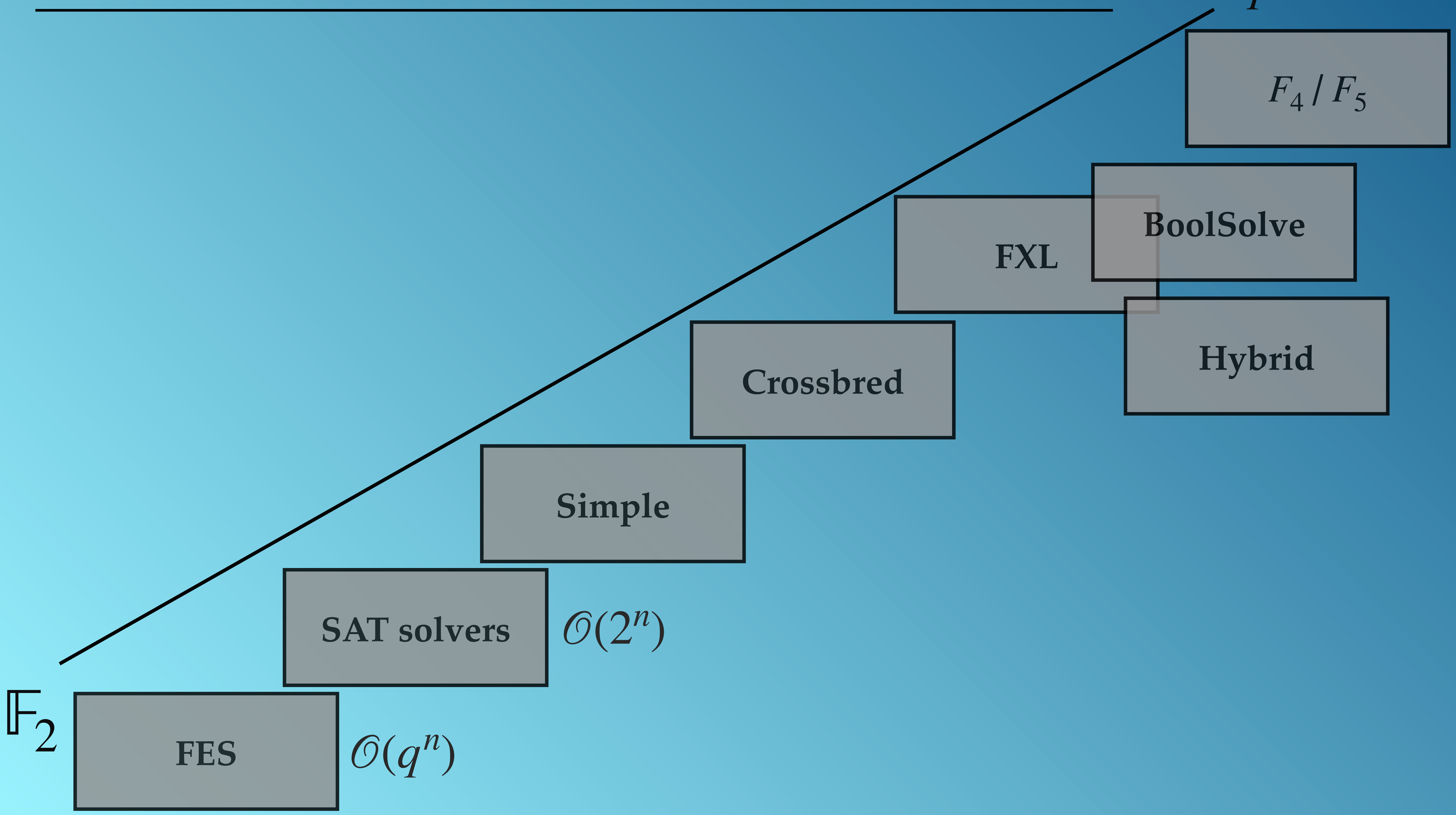
XOR-enabled SAT solvers: take as input XOR constraints as well; perform Gaussian elimination;
*CryptoMiniSat, WDSat

Overview of solvers



Macaulay matrix

Linearisation

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f_5 &: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
f_1 &: y_2 + y_5 + x_1 + x_3 + x_4 = 0\\
f_2 &: y_4 + y_3 + y_6 + x_1 + x_2 + x_4 = 0\\
f_3 &: y_5 + y_6 + x_1 + x_3 + 1 = 0\\
f_4 &: y_1 + y_2 + y_4 + x_3 + x_4 + 1 = 0\\
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\end{aligned}
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Linearisation

Linearisation adds solutions: a *random* quadratic system of m equations in n variables, when $n = m$, is expected to have one solution (probability is $\sim \frac{1}{q}$ for systems over $\mathbb{F}_q$). The corresponding linearised system has a solution space of dimension $\binom{n+1}{2} - m$.

$\binom{n}{2}$ quadratic plus n linear monomials

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$\binom{n}{2}$ quadratic plus n linear monomials

Loss of information: e.g. assignment $x_1 = 1; x_2 = 0; y_1 = 1;$ is part of a valid solution to the linearised system, but $x_1 x_2 \neq y_1$.

Macaulay matrix

Monomials →

Equations ↓

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1											
f_2											
f_3											
f_4											
f_5											
f_6											

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f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

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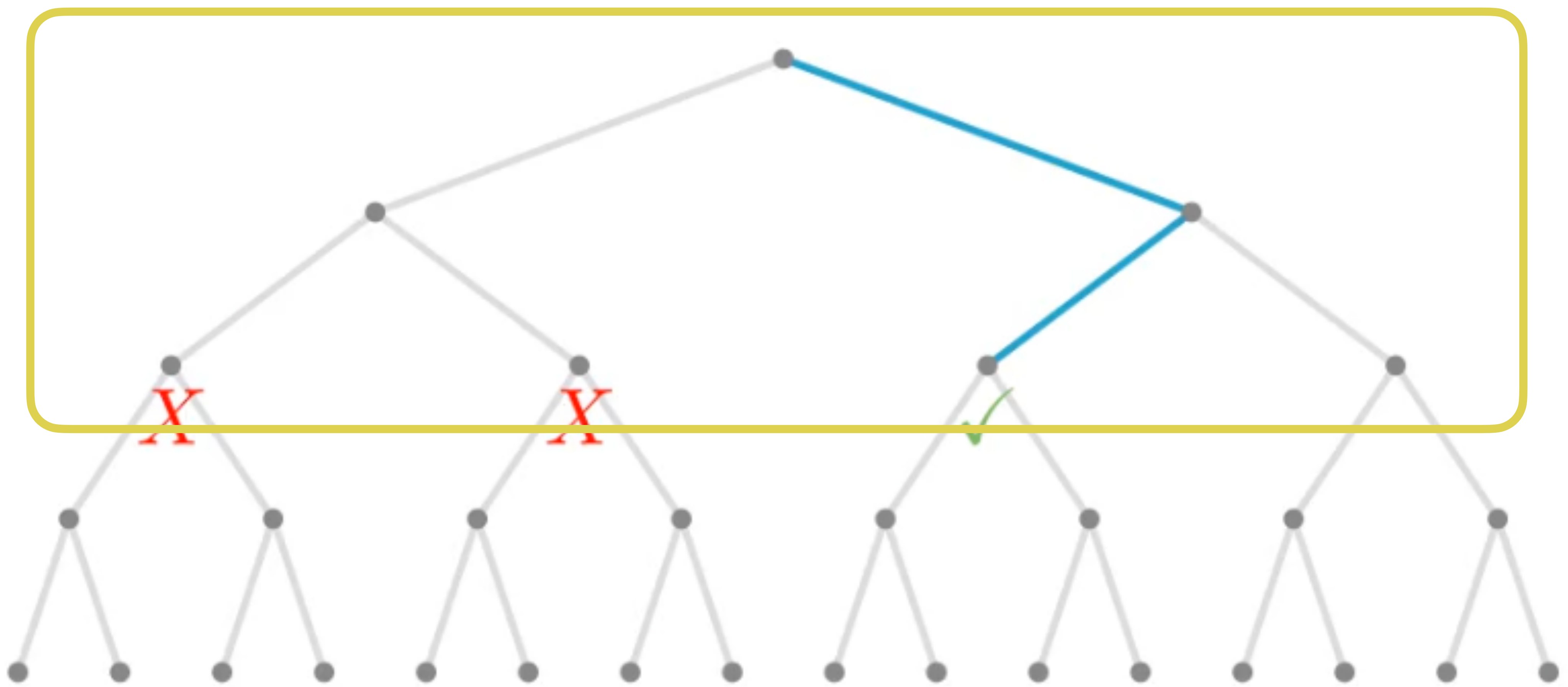
Simple algorithm

[Bouillaguet, Delaplace, T., 2021]

Simple algorithm

- Partial assignment
  - Gaussian elimination



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Simple algorithm

Guess sufficiently many variables so that the remaining polynomial system can be solved by linearization.

Simple algorithm: complexity

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- n - number of variables
- m - number of equations

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$$\mathcal{O}(2^{n-\sqrt{2m}})$$

Simple algorithm: complexity

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- m - number of equations

↳ Enumeration ends when:

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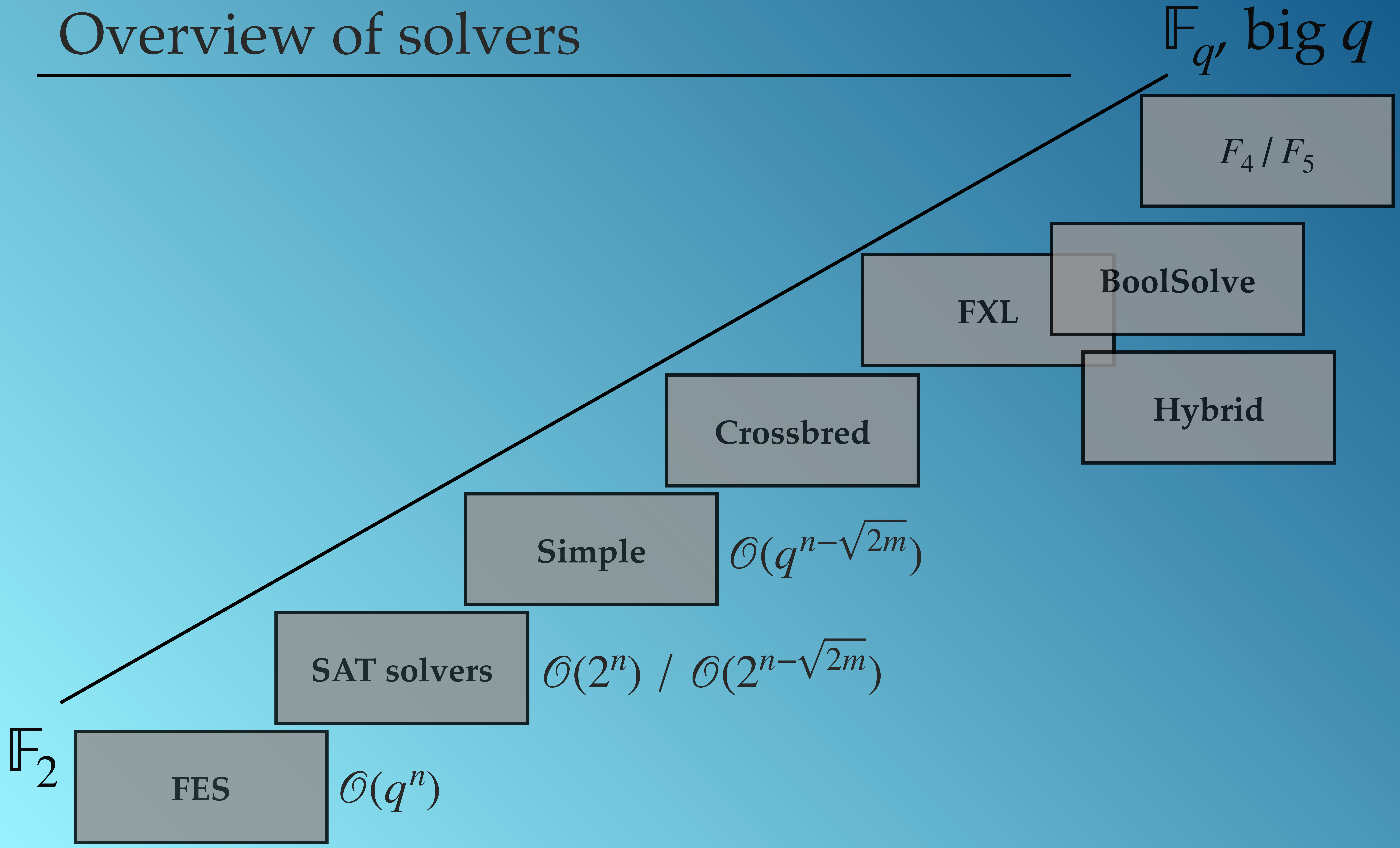
↳ $\mathcal{O}(2^{n-\sqrt{2m}})$

→ See also: Quantum BDT [Edme, Fouque, Schrottenloher]

Overview of solvers



Gröbner basis algorithms

[Buchberger, 1965]
[Lazard, 1983]
F_4/F_5 [Faugère, 1999/2002]
(XL [Courtois, Klimov, Patarin, Shamir, 2000])

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

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$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

$D = 3$

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0				
f_2	0	0	1	1	1	0	1	1	0	1	0				
f_3	0	0	0	1	0	1	0	1	1	0	1				
f_4	1	1	0	1	1	0	0	0	1	1	1				
f_5	1	0	1	1	1	0	0	0	1	0	0				
f_6	0	1	1	1	0	0	1	1	1	1	0				
x_1f_1															
x_2f_1															
$\dots$															

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$
$$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$$
$$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$$
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$$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$$
$$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$$

$D = 4$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
x_2f_1																
… $x_1x_2f_1$																

$x_1x_3f_1$

Gröbner basis

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- Let $R = \mathbb{F}_q[x_1, \ldots, x_n]$ be the polynomial ring in n variables.

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- By the Nullstellensatz: $\mathbf{I}(V(I)) = I$, where $\mathbf{I}(V)$ denotes the ideal of V, i.e. $\mathbf{I}(V) = \{f \in R \,|\, f(a) = 0 \text{ for all } a \in V\}$ (Similar to Gauss' fundamental theorem, but for polynomials in many variables).

Gröbner basis

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Example. The shape of a GB with respect to the lexicographic order

$$f_1 : x_1x_3 + x_1 + x_2x_4 + x_5 + x_6 + 1 = 0$$
$$f_2 : x_1x_4 + x_1 + x_2x_3 + x_2 + x_3x_4 + x_3x_6 + x_4 + x_5 = 0$$
$$f_3 : x_1x_5 + x_1 + x_2 + x_3x_4 + x_6 + 1 = 0$$
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$$
\begin{aligned}
f_1 &: x_1x_3 + x_1 + x_2x_4 + x_5 + x_6 + 1 = 0\\
f_2 &: x_1x_4 + x_1 + x_2x_3 + x_2 + x_3x_4 + x_3x_6 + x_4 + x_5 = 0\\
f_3 &: x_1x_5 + x_1 + x_2 + x_3x_4 + x_6 + 1 = 0\\
f_4 &: x_1x_2 + x_1x_3 + x_2x_5 + x_3 + x_4 + x_6 + 1 = 0\\
f_5 &: x_1x_4 + x_2x_3 + x_2x_5 + x_5x_6 + 1 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_1 + x_2 + x_3x_6 + x_3 + x_5 = 0
\end{aligned}
\longrightarrow
\begin{aligned}
f'_1 &: x_1 + x_6 = 0\\
f'_2 &: x_2 + x_6 = 0\\
f'_3 &: x_3 + x_6 = 0\\
f'_4 &: x_4 + x_6 + 1 = 0\\
f'_5 &: x_5 = 0
\end{aligned}
$$

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*

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f_2 &: x_1x_4 + x_1 + x_2x_3 + x_2 + x_3x_4 + x_3x_6 + x_4 + x_5 = 0 \\
f_3 &: x_1x_5 + x_1 + x_2 + x_3x_4 + x_6 + 1 = 0 \\
f_4 &: x_1x_2 + x_1x_3 + x_2x_5 + x_3 + x_4 + x_6 + 1 = 0 \\
f_5 &: x_1x_4 + x_2x_3 + x_2x_5 + x_5x_6 + 1 = 0 \\
f_6 &: x_1x_3 + x_1x_4 + x_1 + x_2 + x_3x_6 + x_3 + x_5 = 0
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
f_1' &: x_1 + x_6 = 0 \\
f_2' &: x_2 + x_6 = 0 \\
f_3' &: x_3 + x_6 = 0 \\
f_4' &: x_4 + x_6 + 1 = 0 \\
f_5' &: x_5 = 0
\end{aligned}
$$

*****
****
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**
*

$$V(< f_1, \ldots, f_6 >) = \{(0,0,0,1,0,0), (1,1,1,0,0,1)\}$$

Gröbner basis algorithms:

Buchberger, Lazard, F4, F5

↪ Follow the core idea that we described, but combine the equations in an organised way, rather than multiplying them by all possible monomials.

Not covered in this talk:

- Monomial orders
- S-polynomials
- Polynomial long division
- Row reduction in parallel
- Reductions to zero
- Syzygy criterion
- …

XL/Gröbner basis algorithms: complexity

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$$\mathcal{O}\left(mD_{reg}\binom{n+D_{reg}-1}{D_{reg}}^{\omega}\right)$$

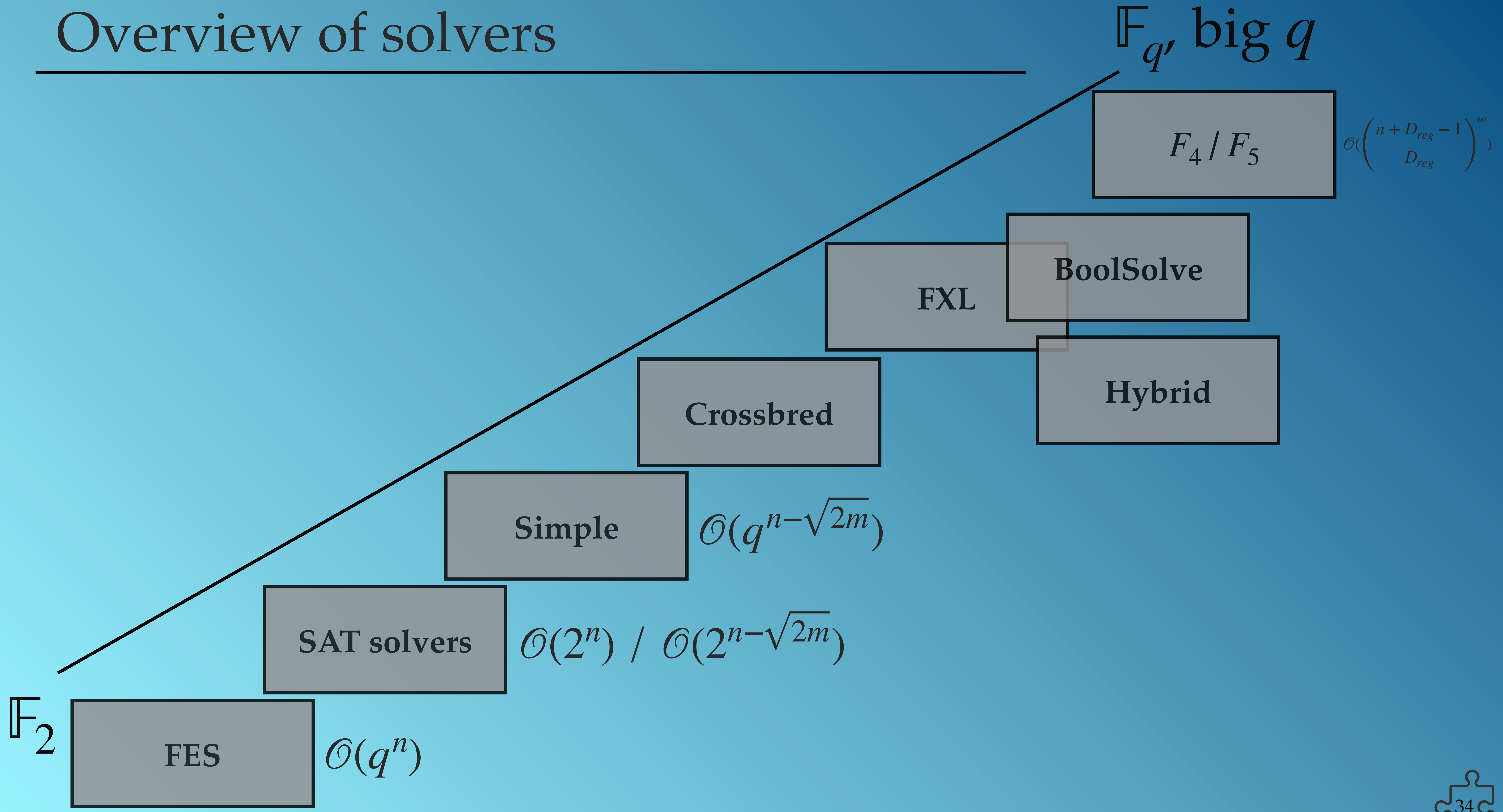
XL/Gröbner basis algorithms: complexity

$$\mathcal{O}\left(mD_{reg} \binom{n + D_{reg} - 1}{D_{reg}}^{\omega} \right)$$

D_{reg}: degree of regularity

↪ the power of the first non-positive coefficient in the expansion of $$\frac{(1-t^2)^m}{(1-t)^n}$$

Overview of solvers



FXL

[Courtois, Klimov, Patarin, Shamir, 2000]

Hybrid

[Bettale, Faugère, Perret, 2009]

BoolSolve

[Bardet, Faugère, Salvy, Spaenlehauer, 2013]

FXL, Hybrid, BoolSolve

Techniques are already covered in the previous section.

Algorithms will be explained in the summary.

The crossbred algorithm

[Joux, Vitse, 2017]

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
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$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

Crossbred algorithm

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$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Put matrix in reduced row echelon form

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
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f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

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$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Take linear subsystem

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

…if we had another 4 equations

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
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	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

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$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Subsystem is linear in variables $\{x_1, x_2, x_3\}$.

→ Enumerating x_4 will result in a linear subsystem.

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

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	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

$\dots$

Crossbred algorithm

→ Subsystem can be linearised

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$

$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$

$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$

$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$

$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$

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Crossbred algorithm

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→ Subsystem can be linearised

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
…											

…if we had another 4 equations, the subsystem would have a unique solution.

Otherwise: check candidate solutions against the other equations.

Crossbred algorithm

Parameters of the algorithm: D, k, d, h

- → Enumerate h variables.
- → Choose k of the remaining variables.
- → Augment system up to degree D (compute degree-D Macaulay matrix).
- → Take the subsystem that is at most degree d in the k chosen variables.
- → Enumerate all but the k chosen variables.
- → Linearise the subsystem and solve it.
- → Check if candidate solutions are consistent with the rest of the system.

Crossbred algorithm

Parameters of the algorithm: D, k, d, h

- Enumerate h variables.
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- Take the subsystem that is at most degree d in the k chosen variables.
- Enumerate all but the k chosen variables.
- Linearise the subsystem and solve it.
- Check if candidate solutions are consistent with the rest of the system.

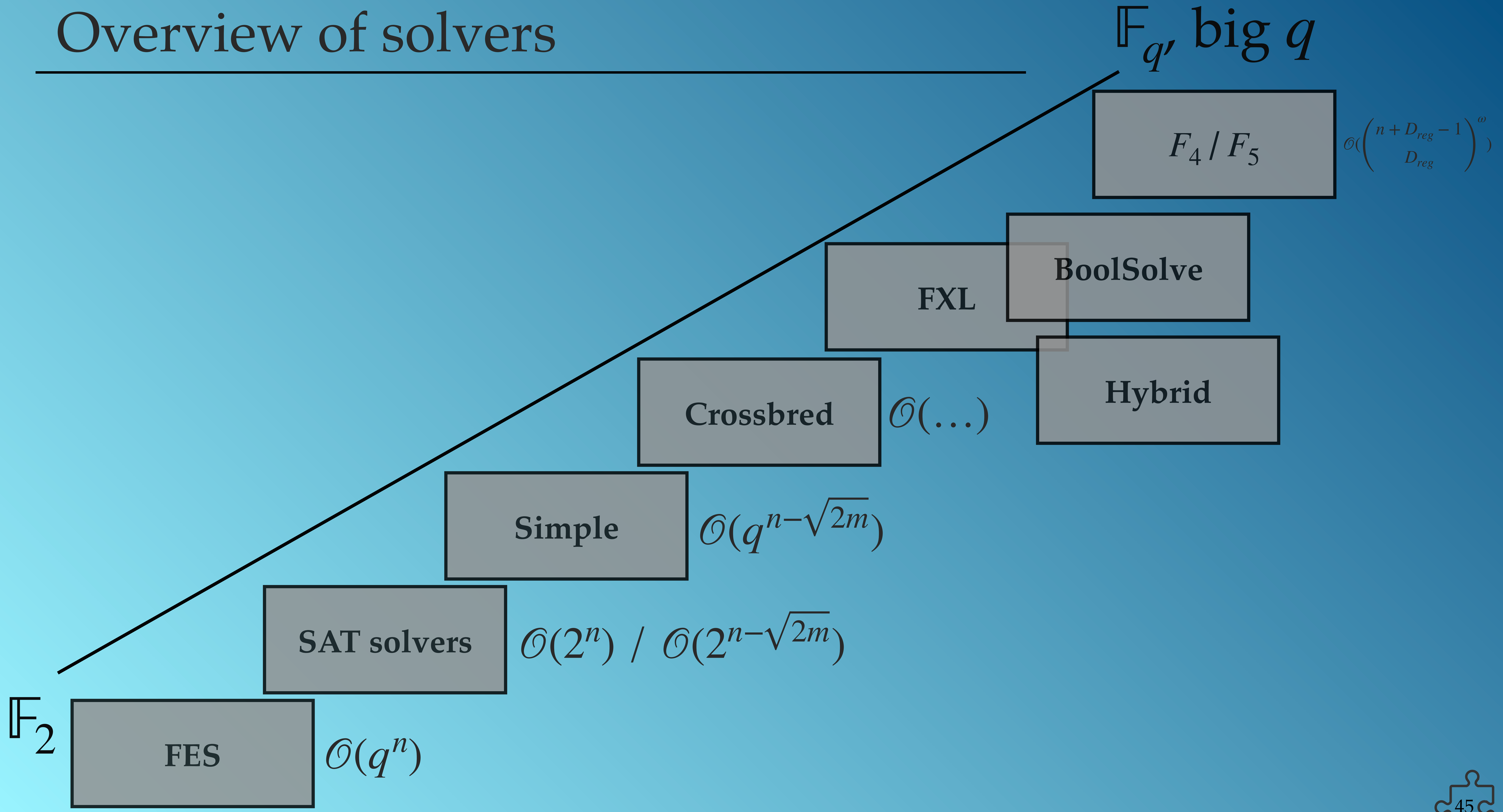
The complexity is calculated as the best trade-off between the four parameters.

Crossbred algorithm

	Number of Variables (n)	Seed (0,1,2,3,4)	Date	Contestants	Computational Resource	Data
1	83	0	2023/09/16	Charles Bouillaguet and Julia Sauvage	https://gitlab.lip6.fr/almasty/hpXbred, 3488 AMD EPYC 7J13 cores on the Oracle public cloud	Details
6	74	0	2016/12/17	Antoine Joux	New hybridized XL related algorithm, Heterogeneous cluster of Intel Xeon @ 2.7-3.5 Ghz	Details
7	74	4	2017/11/15	Kai-Chun Ning, Ruben Niederhagen	Parallel Crossbred, 54 GPUs in the Saber cluster	Details
25	66	0	2016/01/22	Tung Chou, Ruben Niederhagen, Bo-Yin Yang	Gray Code enumeration, Rivyera, 128 Spartan 6 FPGAs	Details

Fukuoka MQ challenge record computations ($m = 2n$)

Overview of solvers



Summary

- (Partial) enumeration
- Candidate solutions (subsystem)
- Conflict search
- Extending to higher degrees
- Computing a Gröbner Basis

- **FES**
- **Simple**
- **FXL**
- F_4 / F_5
- **SAT solvers**
- **Crossbred**
- **BoolSolve**
- **Hybrid**

Summary

(Partial) enumeration

Candidate solutions (subsystem)

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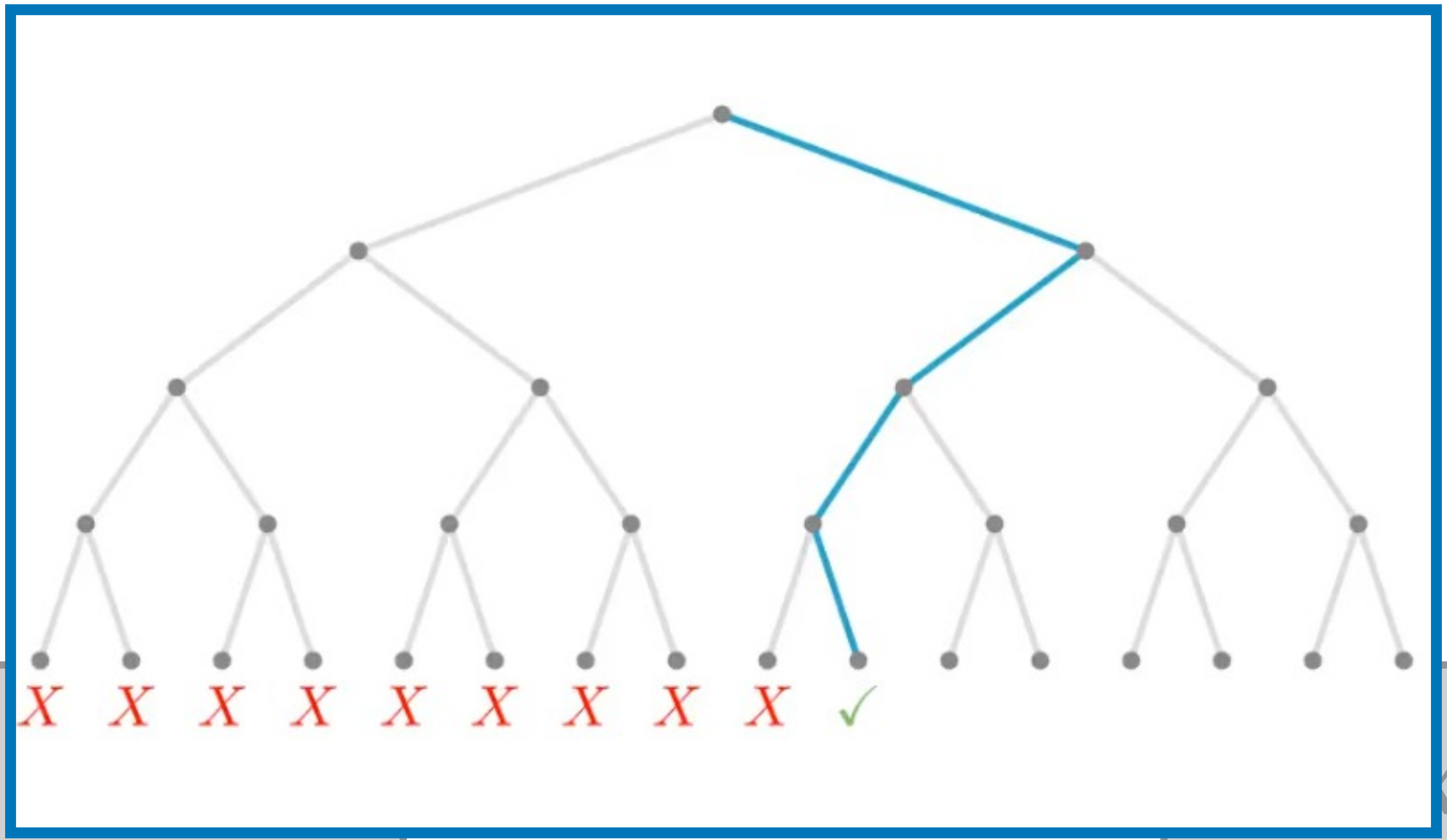
Computing a Gröbner Basis



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XL

F_4 / F_5

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	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
…											

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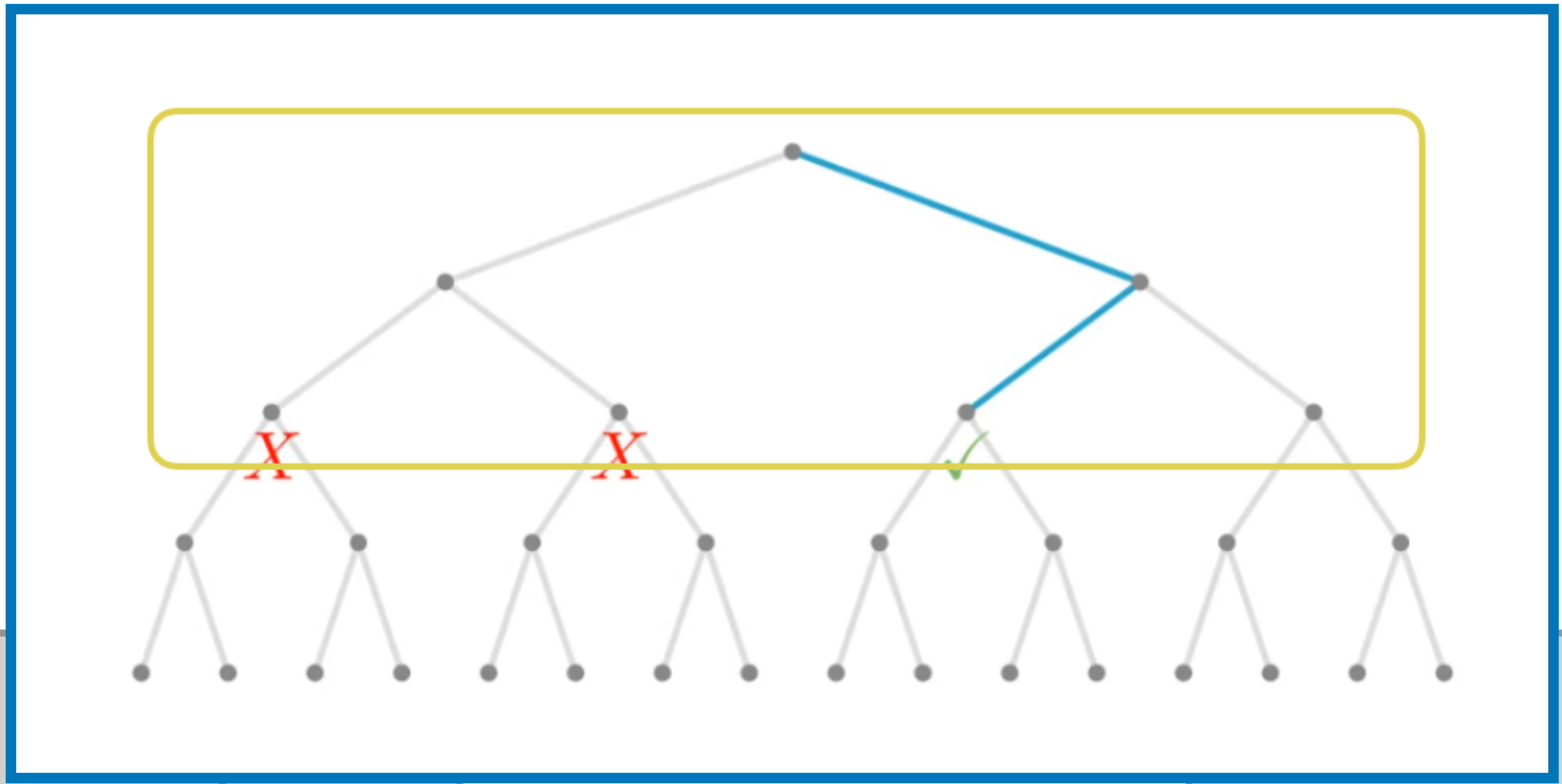

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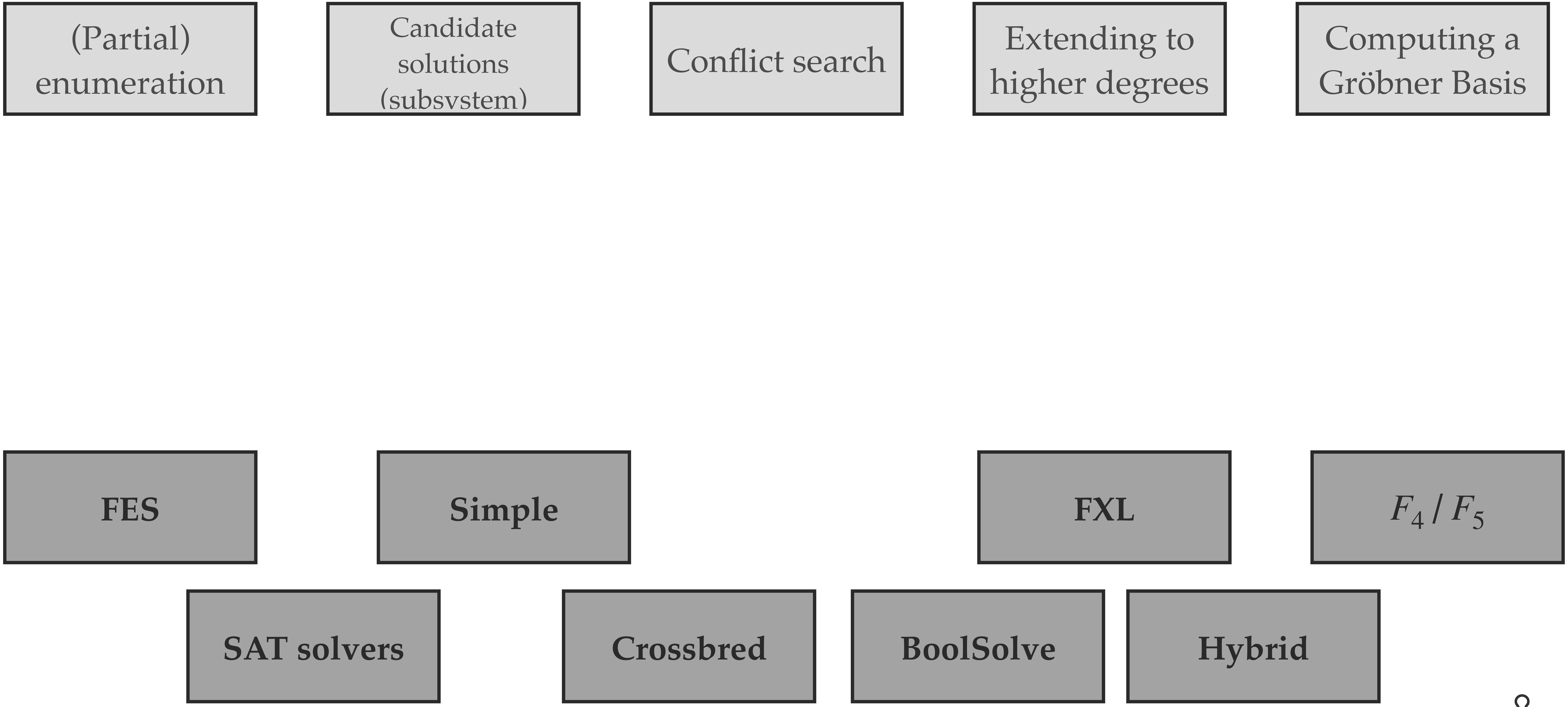
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Candidate solutions (subsystem)
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Extending to higher degrees
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FES
Simple
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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
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…																
$x_1x_2f_1$																
$x_1x_3f_1$																

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Conflict search

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FES

Simple

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F_4 / F_5

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$$f_1' : x_1 + x_6 = 0$$
$$f_2' : x_2 + x_6 = 0$$
$$f_3' : x_3 + x_6 = 0$$
$$f_4' : x_4 + x_6 + 1 = 0$$
$$f_5' : x_5 = 0$$

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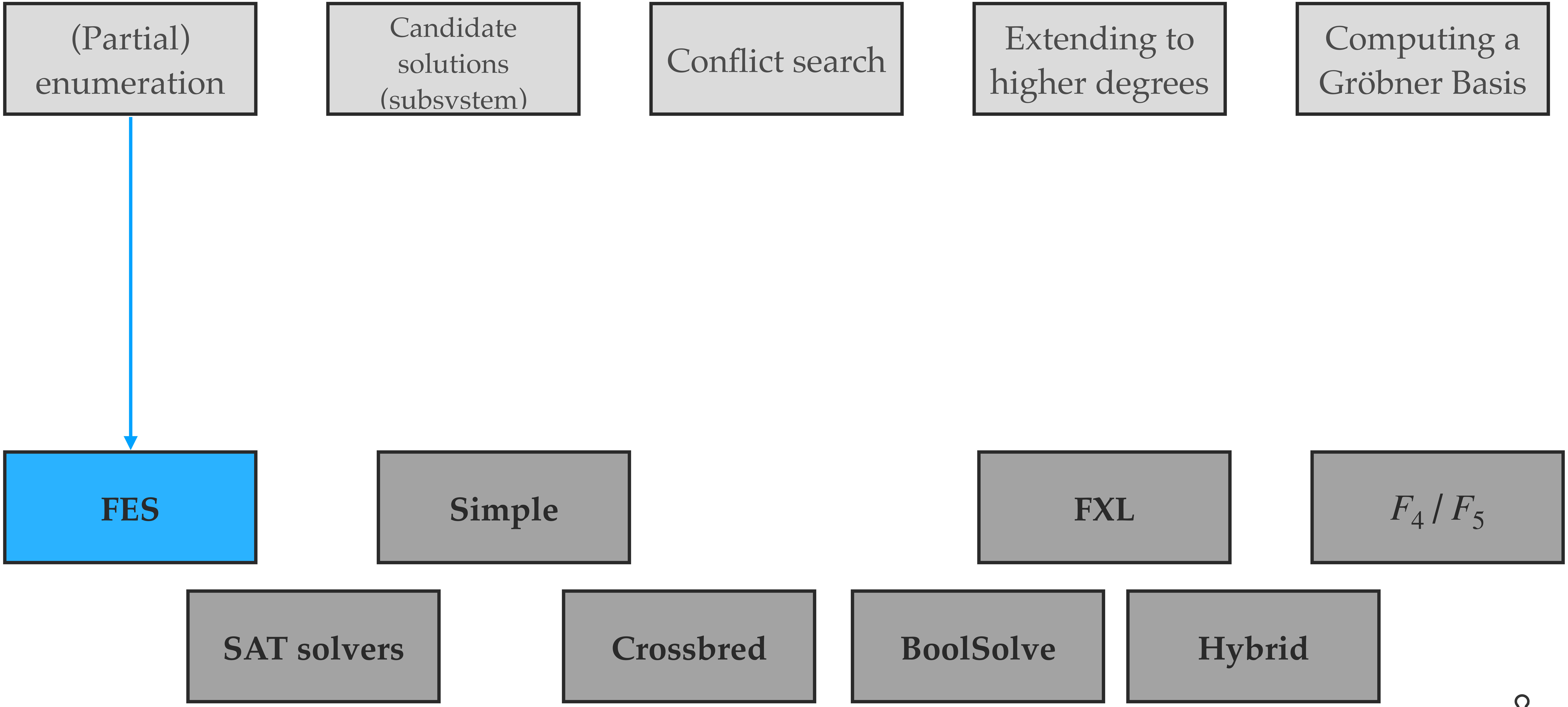

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Simple
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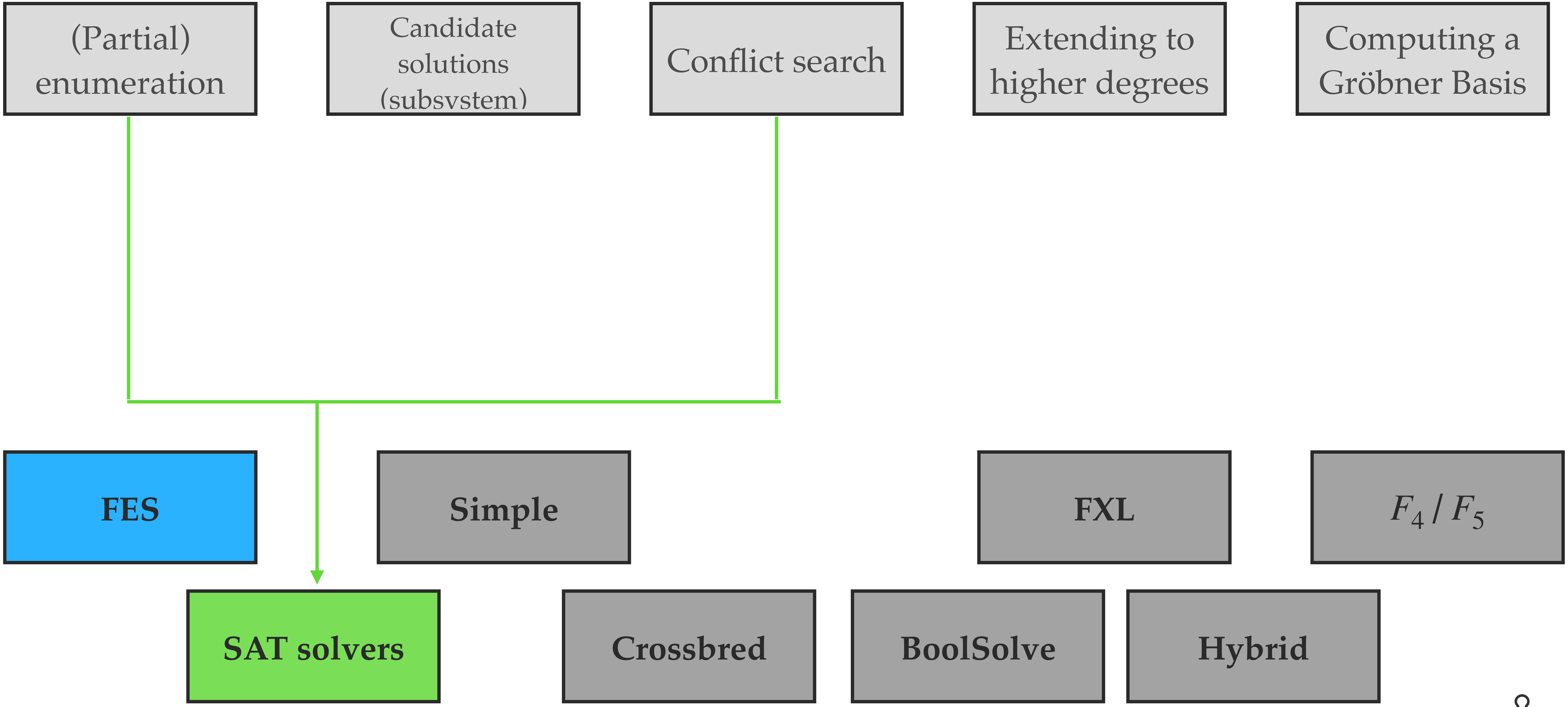

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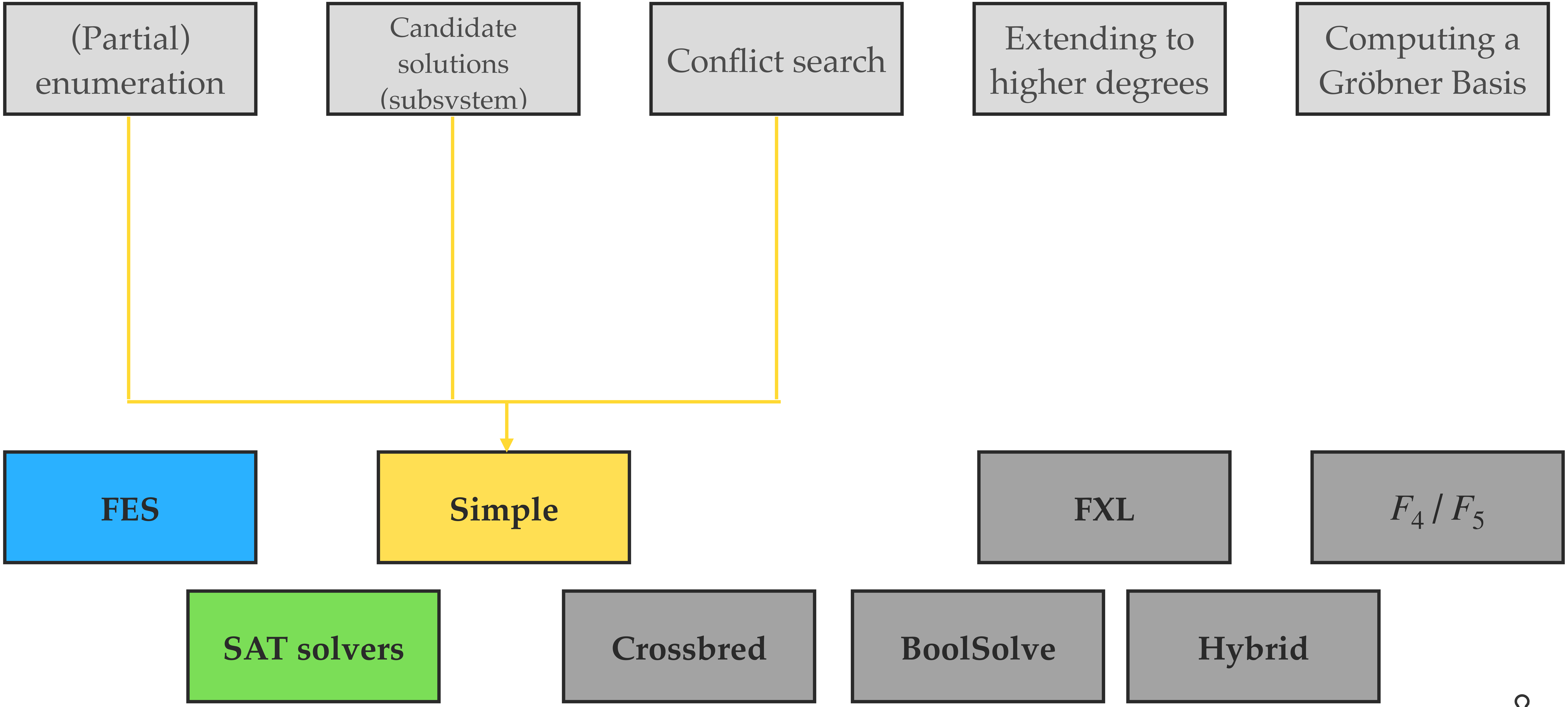
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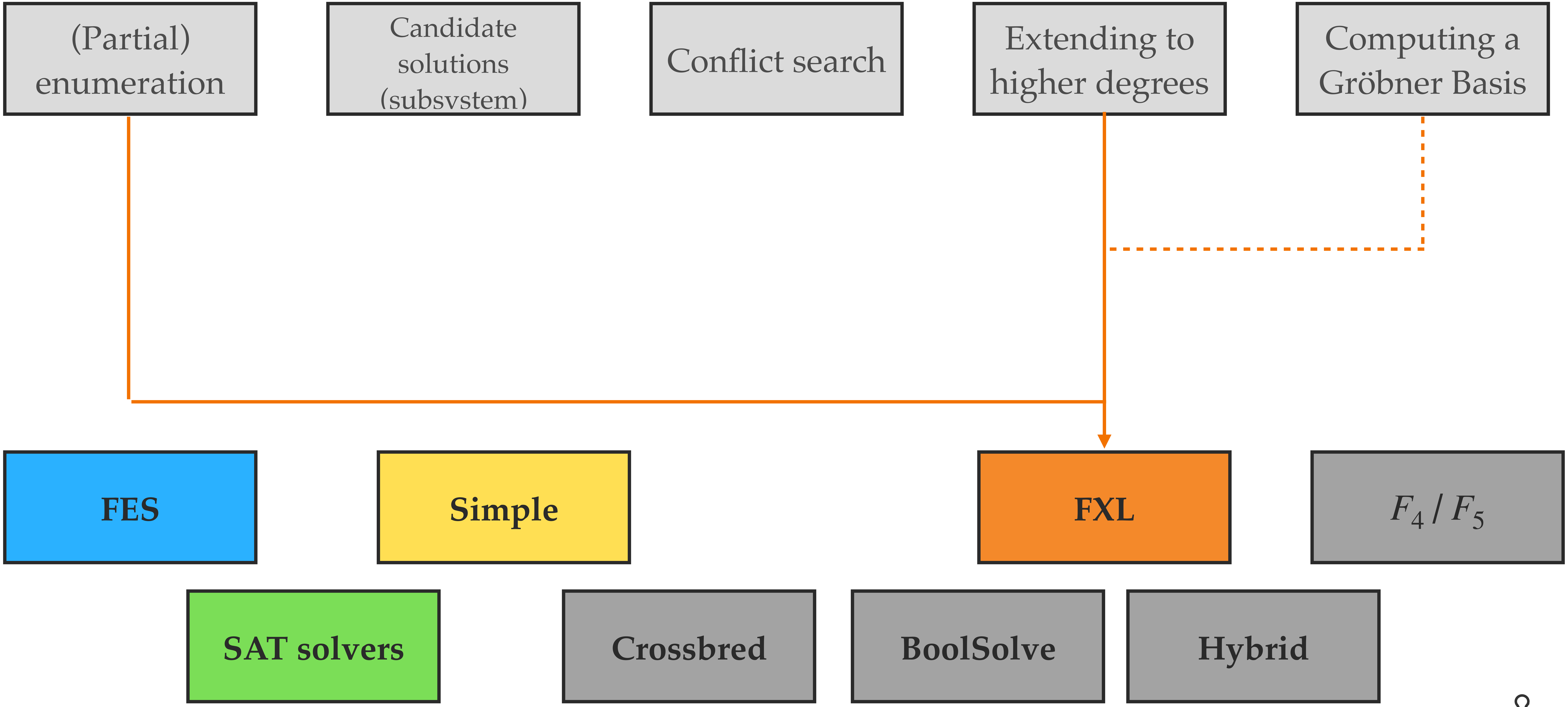
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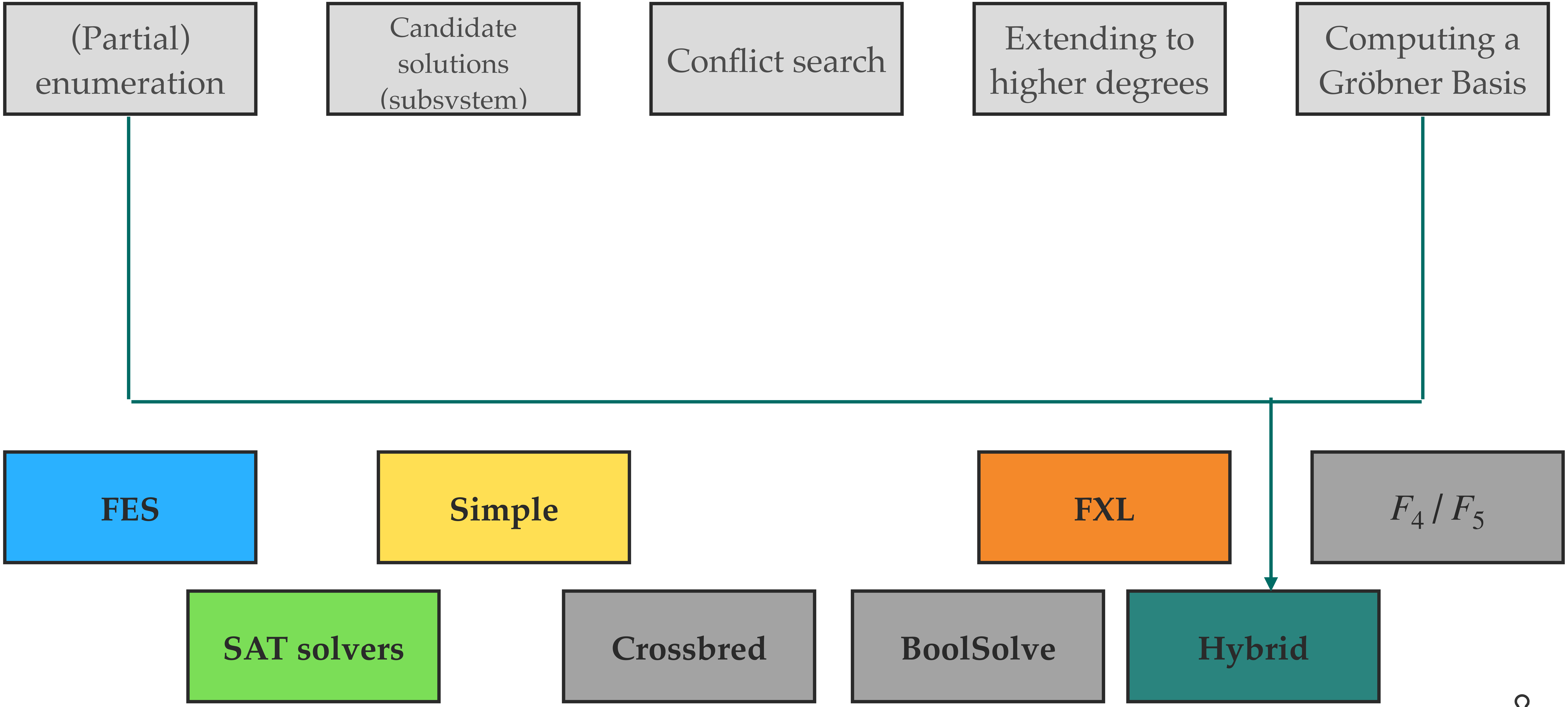

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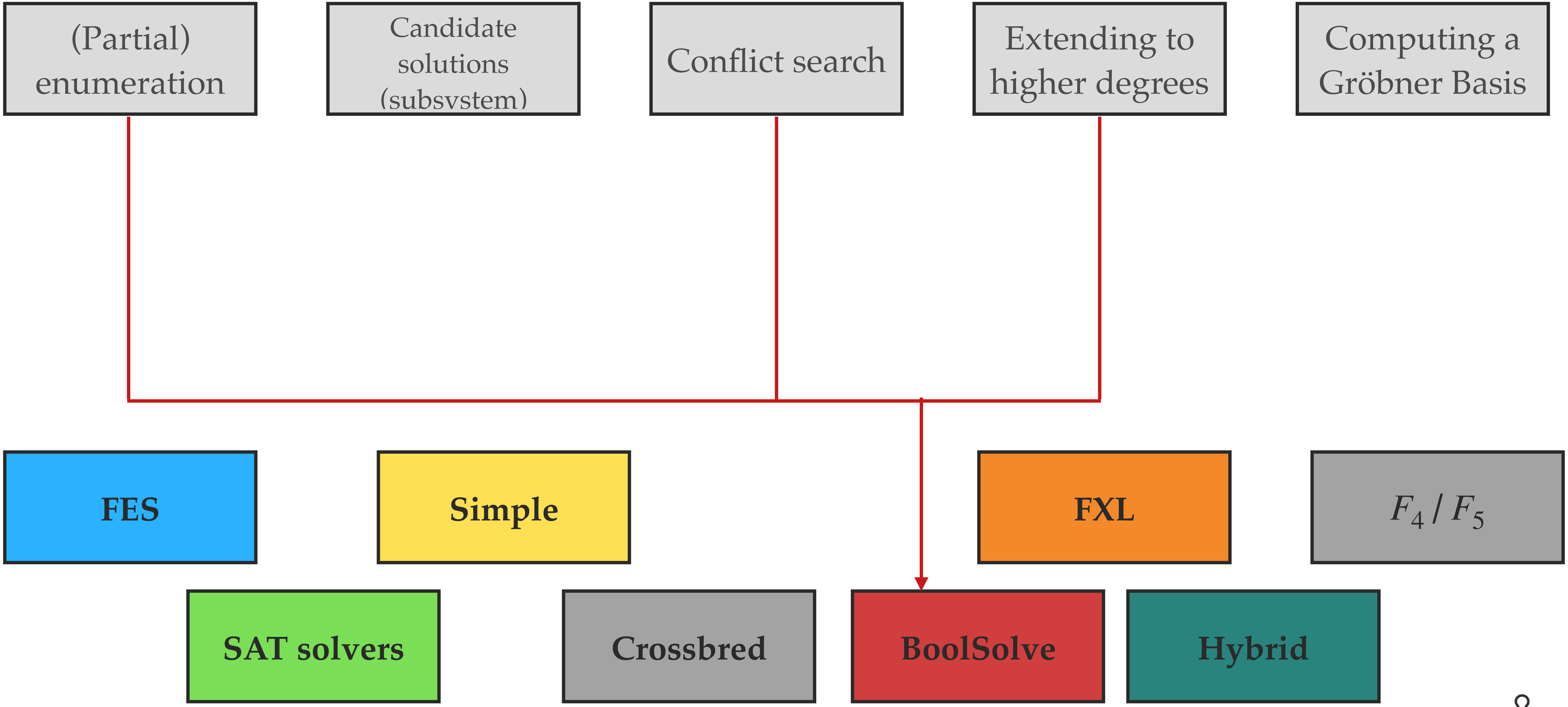
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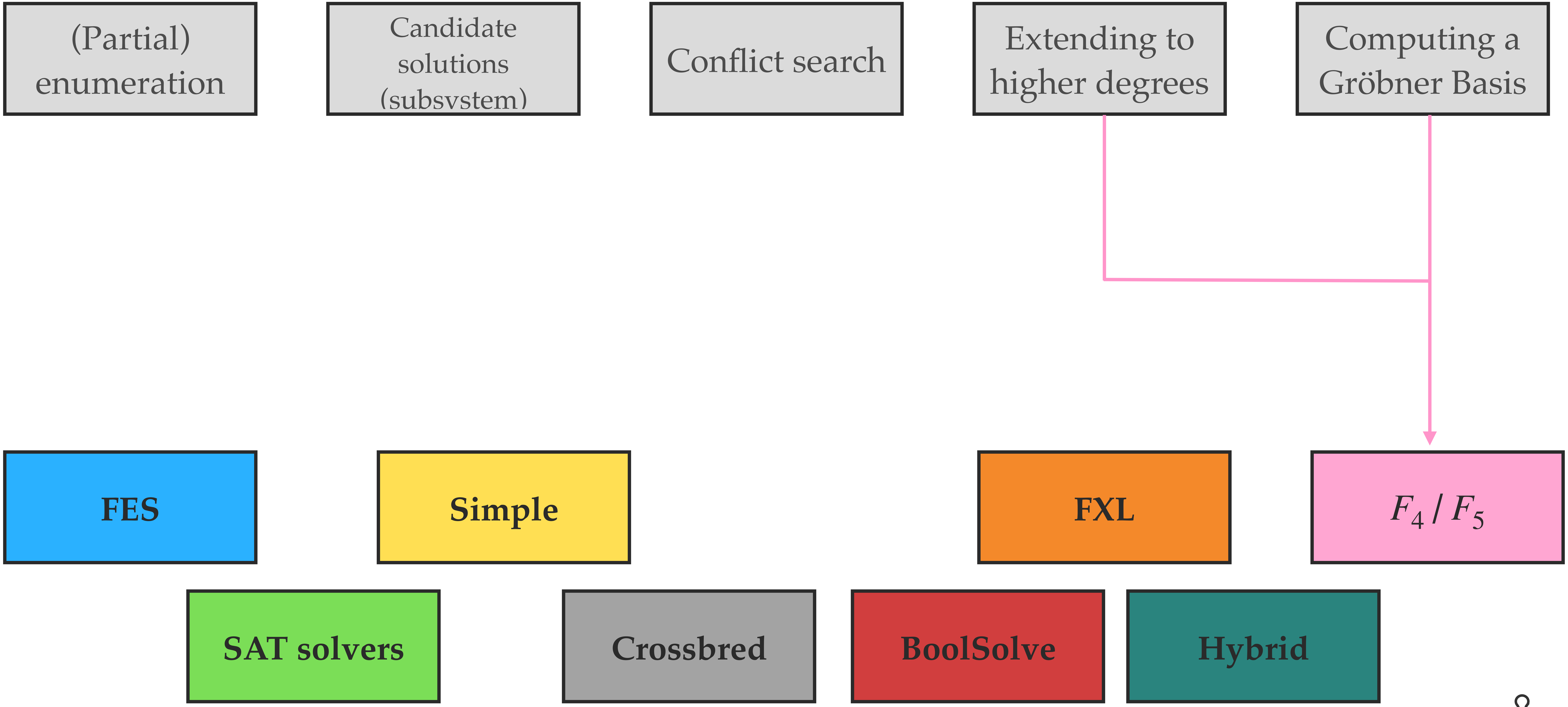
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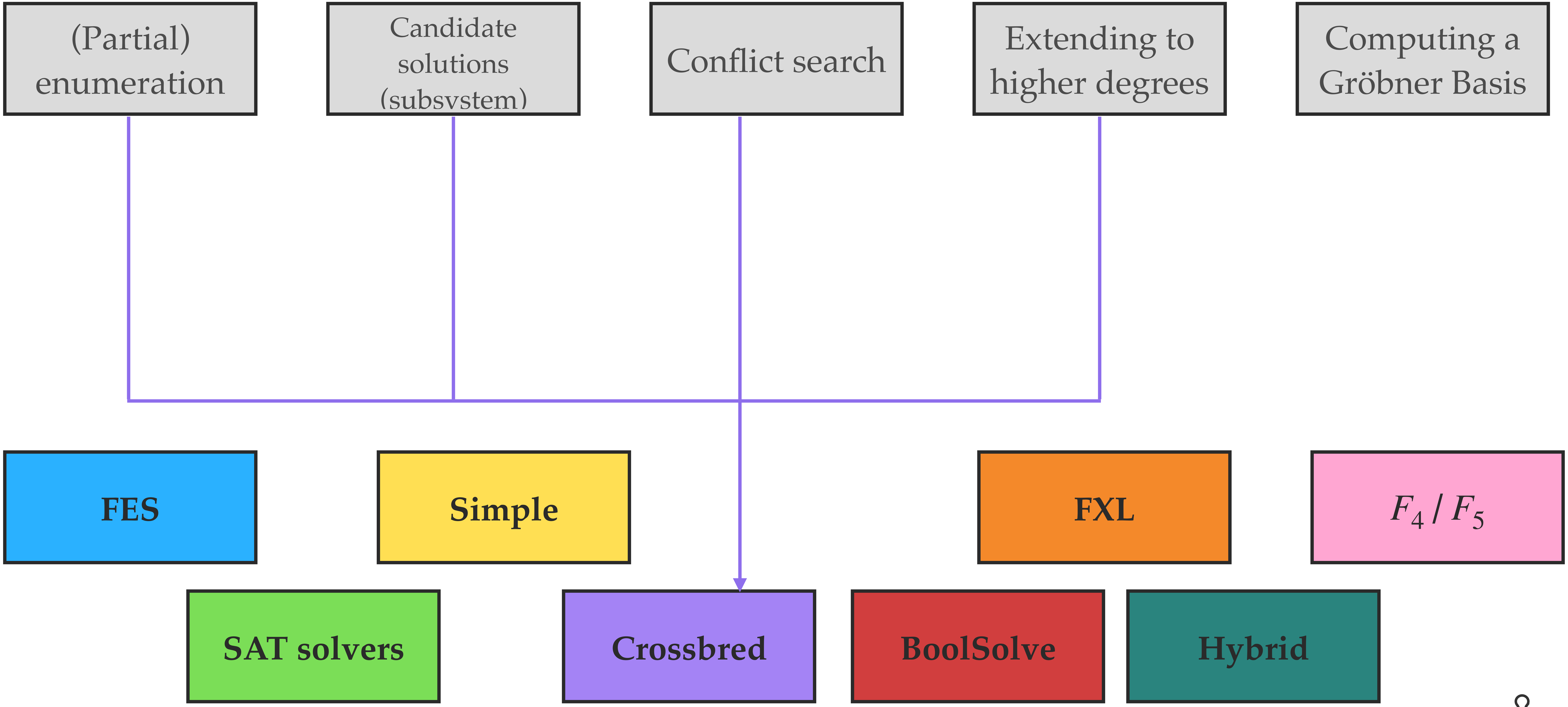
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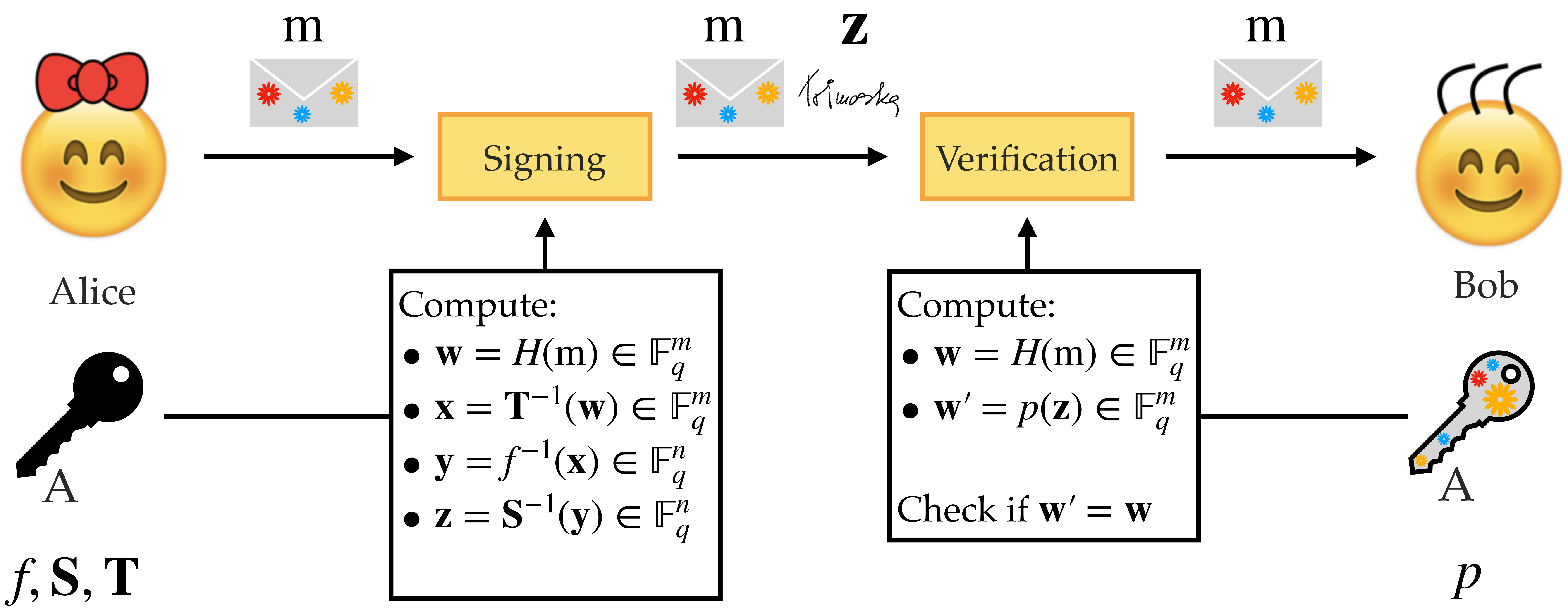
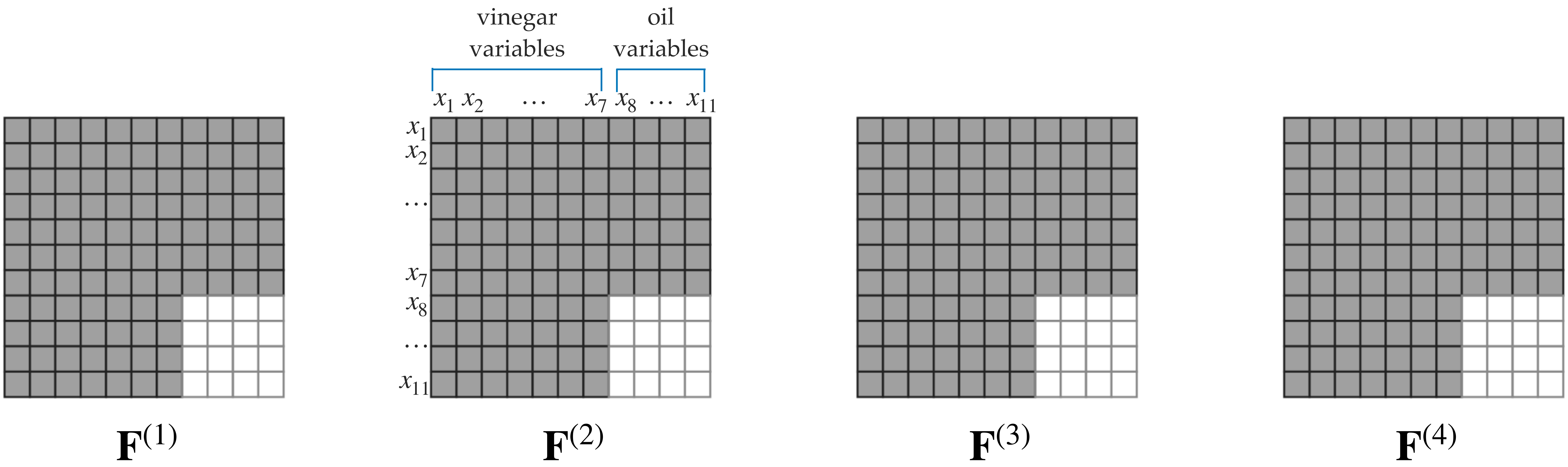
Modelisation: Attacks on UOV

The trapdoor construction (recall)



The UOV central map (recall)

Toy example: $v = 7, m = 4$



*Grayed areas represent the entries that are possibly nonzero; blank areas denote the zero entries;

Attacks on UOV

- Direct attack
- Reconciliation attack
- Kipnis-Shamir attack
- Intersection attack

Direct attack

Direct attack

- Try to forge a signature with only the knowledge of the public key.

Direct attack

Try to forge a signature with only the knowledge of the public key.

Constraint for modelisation

For a target $\mathbf{w}$, find $\mathbf{z}$ such that $p(\mathbf{z}) = \mathbf{w}$.

Direct attack

Try to forge a signature with only the knowledge of the public key.

Constraint for modelisation

For a target $\mathbf{w}$, find $\mathbf{z}$ such that $p(\mathbf{z}) = \mathbf{w}$.

Equations:

$$
\begin{aligned}
\mathbf{z}^\top \mathbf{P}^{(1)} \mathbf{z} &= w_1 \\
\mathbf{z}^\top \mathbf{P}^{(2)} \mathbf{z} &= w_2 \\
&\dots \\
\mathbf{z}^\top \mathbf{P}^{(m)} \mathbf{z} &= w_m
\end{aligned}
$$

Direct attack

Try to forge a signature with only the knowledge of the public key.

Constraint for modelisation

For a target $\mathbf{w}$, find $\mathbf{z}$ such that $p(\mathbf{z}) = \mathbf{w}$.

→ Equations:

$$\mathbf{z}^\top \mathbf{P}^{(1)} \mathbf{z} = w_1$$
$$\mathbf{z}^\top \mathbf{P}^{(2)} \mathbf{z} = w_1$$
$$\dots$$
$$\mathbf{z}^\top \mathbf{P}^{(m)} \mathbf{z} = w_m$$

Reconciliation attack

[Ding, Yang, Chen, Chen, Cheng, 2008]

The secret subspace O

The map p with a UOV trapdoor vanishes on a linear subspace $O \subset \mathbb{F}_q^n$ of $\dim(O) = m$:

$$p(\mathbf{o}) = 0, \text{ for all } \mathbf{o} \in O.$$

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Why ?

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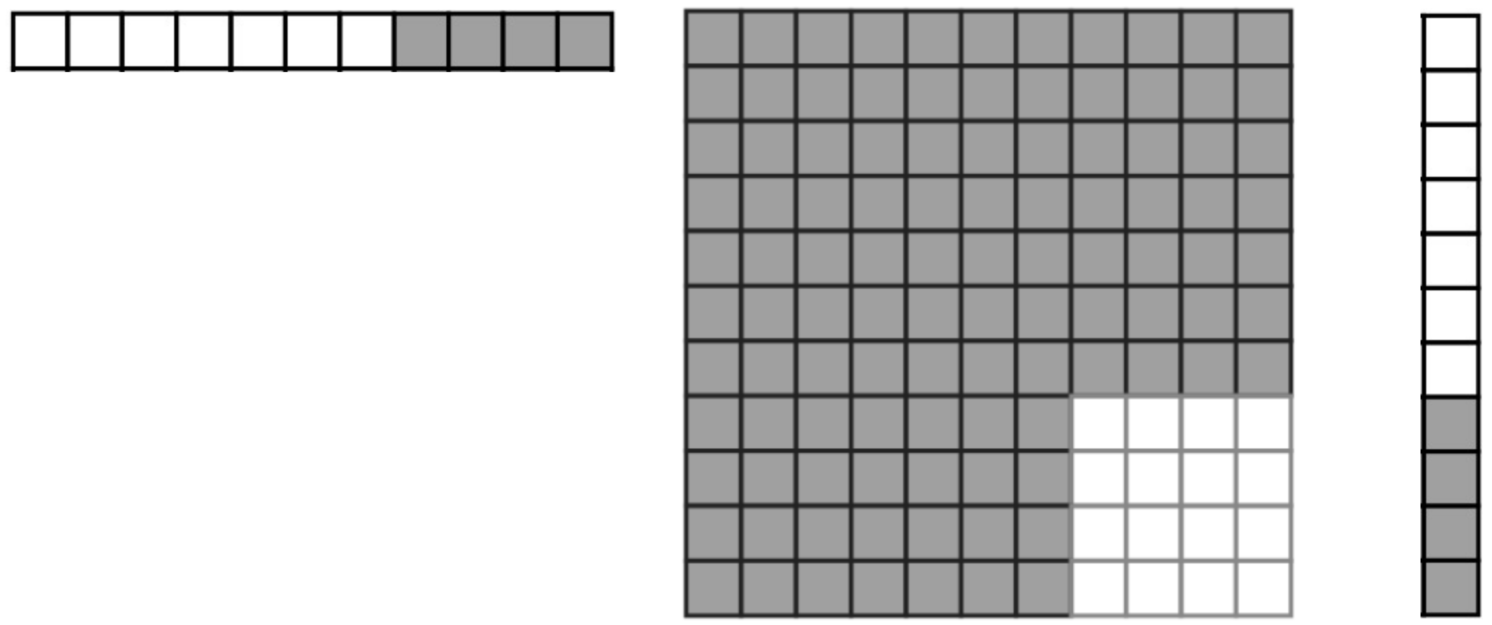
Why ?

Let $O' \in \mathbb{F}_q^n$ be the m-dimensional space that consists of all the vectors whose first $n - m$ entries (corresponding to the vinegar variables) are zero: $O' = \{\mathbf{v} \,|\, v_i = 0 \text{ for all } i \leq n - m\}$.



$= 0$

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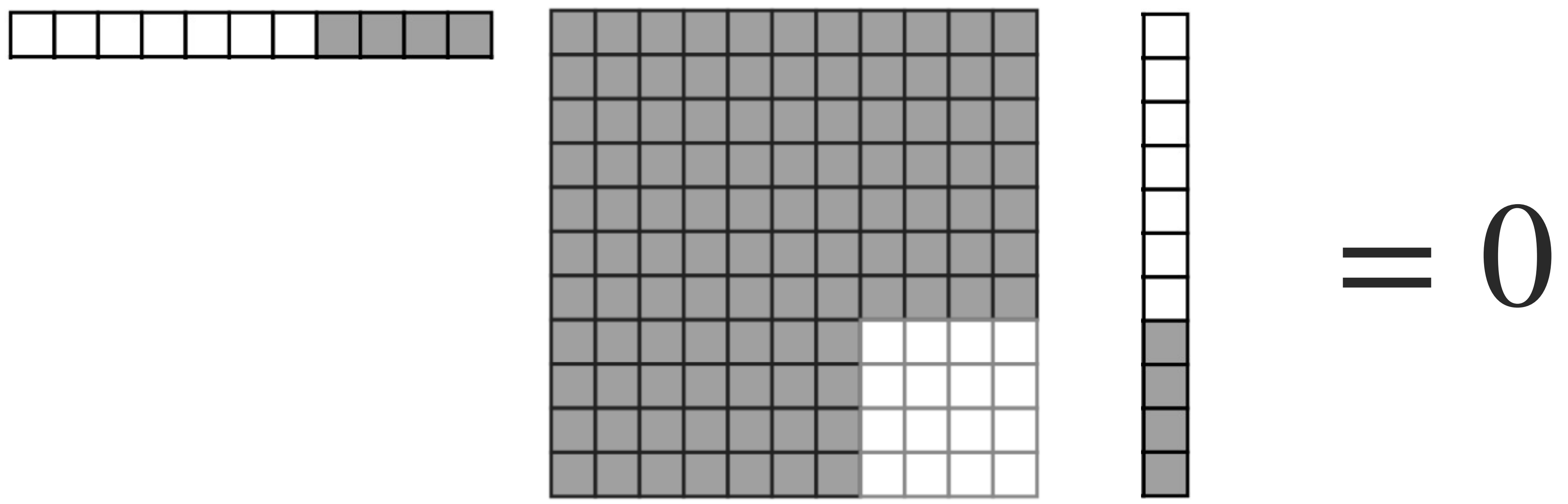

f vanishes on O'.

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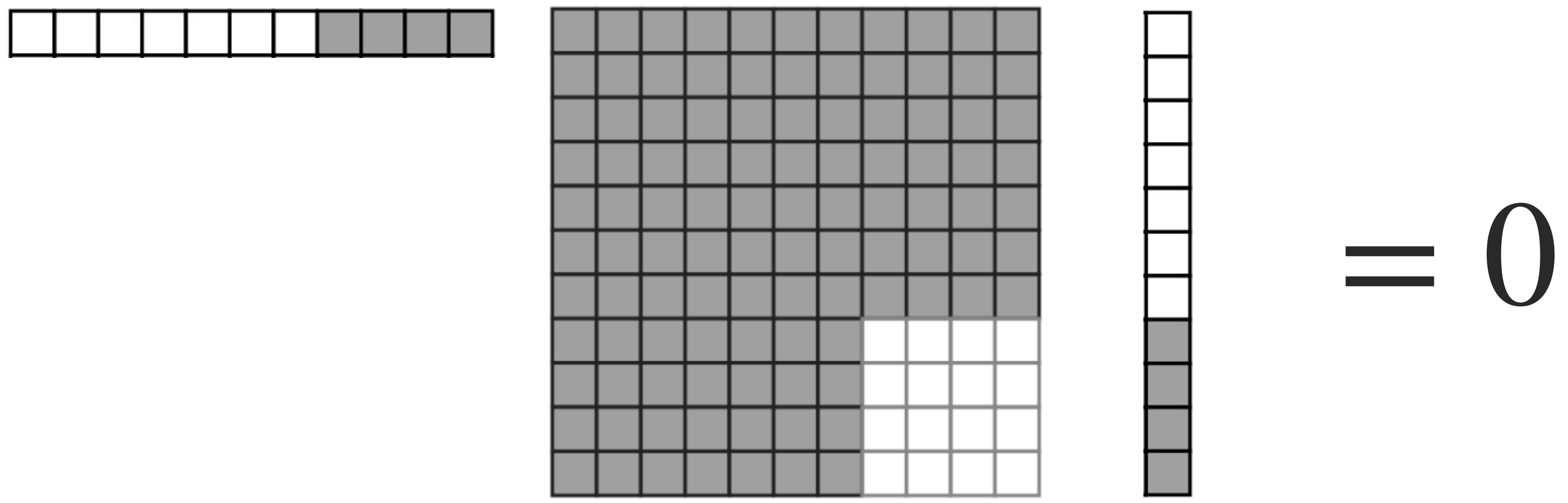

f vanishes on O'.

Let $O = \mathbf{S}^{-1}(O')$.

The secret subspace O

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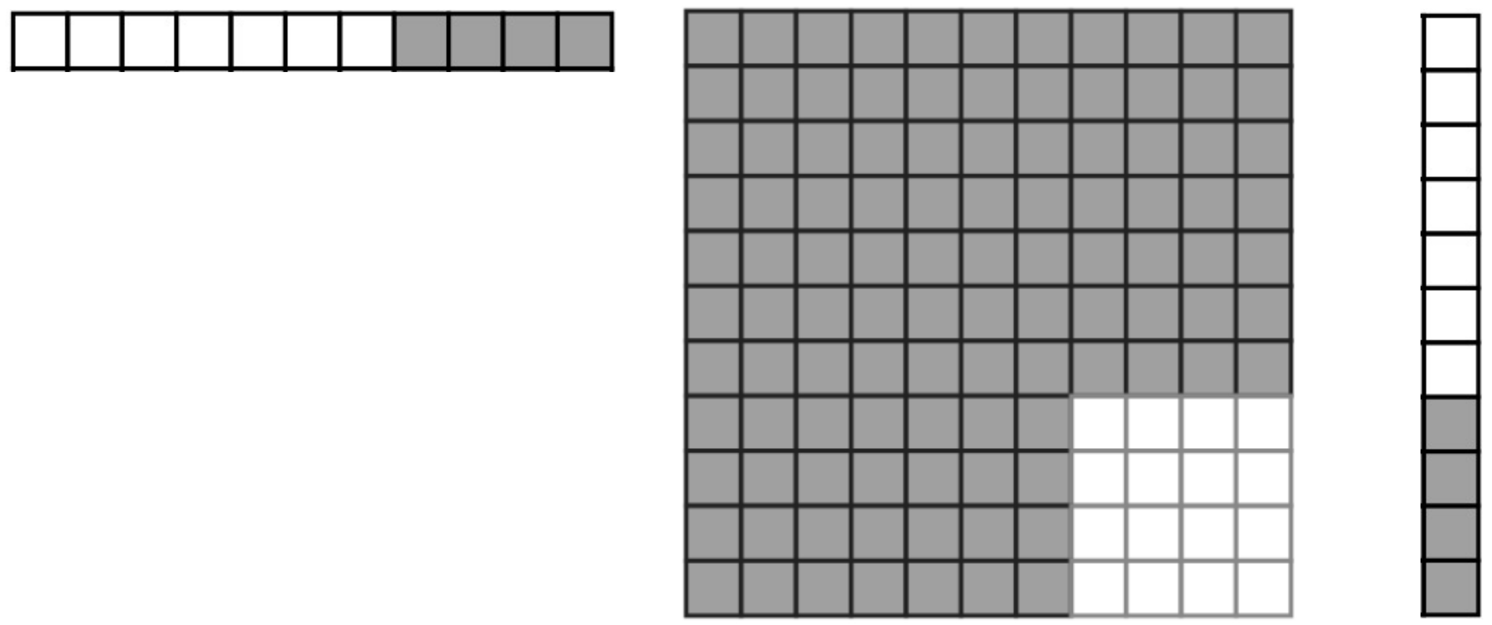

$= 0$

f vanishes on O'.

Let $O = \mathbf{S}^{-1}(O')$.

p vanishes on O.

Reconciliation attack

Find the secret oil subspace O : find m linearly independent vectors in O.

The polar form

The polar form of a quadratic map $p = (p^{(1)}, \ldots, p^{(m)})$ is the bilinear form $p' = (p'^{(1)}, \ldots, p'^{(m)})$ such that

$$p'^{(k)}(\mathbf{x}, \mathbf{y}) = p^{(k)}(\mathbf{x} + \mathbf{y}) - p^{(k)}(\mathbf{x}) - p^{(k)}(\mathbf{y}), \text{ for all } k \in \{1, \ldots, m\} .$$

The polar form

The polar form of a quadratic map $p = (p^{(1)}, \ldots, p^{(m)})$ is the bilinear form $p' = (p^{'(1)}, \ldots, p^{'(m)})$ such that

$$p^{'(k)}(\mathbf{x}, \mathbf{y}) = p^{(k)}(\mathbf{x} + \mathbf{y}) - p^{(k)}(\mathbf{x}) - p^{(k)}(\mathbf{y}), \text{ for all } k \in \{1, \ldots, m\}\,.$$

What does $p^{'(k)}(\mathbf{x}, \mathbf{y})$ look like ?

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What does $p'^{(k)}(\mathbf{x}, \mathbf{y})$ look like ?

Let $\tilde{\mathbf{P}}^{(k)}$ be the upper triangular representation of $p^{(k)}$.

The polar form

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$$\begin{aligned} p'^{(k)}(\mathbf{x}, \mathbf{y}) &= p^{(k)}(\mathbf{x} + \mathbf{y}) - p^{(k)}(\mathbf{x}) - p^{(k)}(\mathbf{y}) \\ &= (\mathbf{x} + \mathbf{y})^\top \tilde{\mathbf{P}}^{(k)}(\mathbf{x} + \mathbf{y}) - x^\top \tilde{\mathbf{P}}^{(k)}\mathbf{x} - y^\top \tilde{\mathbf{P}}^{(k)}\mathbf{y} \\ &= x^\top \tilde{\mathbf{P}}^{(k)}\mathbf{y} + y^\top \tilde{\mathbf{P}}^{(k)}\mathbf{x} \\ &= x^\top (\tilde{\mathbf{P}}^{(k)} + \tilde{\mathbf{P}}^{(k)\top})\mathbf{y} = x^\top \mathbf{B}^{(k)}\mathbf{y} \end{aligned}$$

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→ So, p' is bilinear and symmetric.

Reconciliation attack

Find the secret oil subspace O : find m linearly independent vectors in O.

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Find the secret oil subspace O : find m linearly independent vectors in O.

Constraint for modelisation

For any vector $\mathbf{o}_i \in O$, we have that $\mathbf{o}_i^\top \mathbf{P}^{(k)} \mathbf{o}_i = 0$ for all $k \in \{1, \ldots, m\}$.
For any pair of vectors $\mathbf{o}_i, \mathbf{o}_j \in O$, we have that $\mathbf{o}_i^\top \mathbf{B}^{(k)} \mathbf{o}_j = 0$ for all $k \in \{1, \ldots, m\}$.

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→ Equations:

For $i \in \{1, \ldots, m\}$ **do**

$\mathbf{o}_i = (o_1, \ldots, o_v, 0, \ldots, 1_{n-i+1}, 0, \ldots, 0)$

Model:

$$\mathbf{o}_i^\top \mathbf{B}^{(k)} \mathbf{o}_j = 0, \text{ for } k \in \{1, \ldots, m\} \text{ and } j < i$$

$$\mathbf{o}_i^\top \mathbf{P}^{(k)} \mathbf{o}_i = 0, \text{ for } k \in \{1, \ldots, m\}$$

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In the first iteration, we have only quadratic equations, so this is the bottleneck. Linear constraints facilitate the resolution of a system.

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Kipnis-Shamir attack

[Kipnis, Shamir, 1998]

The orthogonal complement of a subspace

Let $V \subset \mathbb{F}_q^n$. The orthogonal complement of V is $V^\perp$ such that

$$V^\perp = \{\tilde{\mathbf{v}}_i \in \mathbb{F}_q^n \,|\, \langle \mathbf{v}_j, \tilde{\mathbf{v}}_i \rangle = 0, \text{ for all } \mathbf{v}_j \in V\}.$$

If V is m-dimensional, then $V^\perp$ is $(n-m)$-dimensional.

Kipnis-Shamir attack

Find the secret oil subspace O. Works well for the balanced case ($n = 2m$) - the original proposal of OV.

Kipnis-Shamir attack

Find the secret oil subspace O. Works well for the balanced case ($n = 2m$) - the original proposal of OV.

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→ Oil and Vinegar becomes Unbalanced Oil and Vinegar because of this attack.

Intersection attack

[Beullens, 2021]

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→ The attack can be generalised to find a vector in the intersection of more than two subspaces.

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